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The Split-Complex Boundary Algebra $\mathbb{D}$

Canonical uniqueness, countable profinite construction, and the four-atom generator dictionary

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ABSTRACT

We establish the split-complex boundary algebra $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2 - 1)$ as the canonical scalar algebra of Category τ, giving a countable, constructive, Lean-friendly foundation for boundary characters and identifying $\mathbb{D}$ as the unifying algebraic home of the three companion *hinge papers* — Hyperfactorization, Prime Polarity, and the Master Constant ι_τ — of the Panta Rhei 2nd-Edition bundle. Concretely: (i) we construct the countable profinite boundary ring $\mathcal{R}_\partial = \varprojlim_k \mathbb{Z}/M_k\mathbb{Z}$ via a canonical primorial ladder of moduli M_k, with all compatibility conditions expressed by finite witness predicates; (ii) we prove a fully constructive Chinese remainder theorem and define Teichmüller-style lifts as primitive recursive normalisers; (iii) we identify a unique crossing mediator $\iota_\tau = 2/(\pi + e) \approx 0.341304$ as the canonical balanced element at the lemniscate junction; (iv) we prove a *canonical uniqueness theorem*: any commutative $\mathcal{R}_\partial$-algebra satisfying four explicit τ-kernel structural constraints (binary dimensionality, commutativity, two nontrivial orthogonal idempotents, polarity-swap involution) is canonically isomorphic to $\mathbb{D}$, with the isomorphism forced on generators by the constraints; (v) we establish the *four-atom spectral dictionary* $\{0, e_+, e_-, 1\} \leftrightarrow \{\alpha\text{-null}, \gamma, \eta, \alpha\text{-total}\}$, with ι_τ as the ω-crossing mediator, and the explicit force-mapping correspondence of the Panta Rhei 4+1 sector structure; (vi) we prove the *elliptic complex exclusion theorem* ruling out $\mathbb{Z}[i]/(i^2 + 1)$ as a host algebra, showing that no B/C idempotent decomposition is available in the elliptic setting and clarifying why elliptic complex structures are non-ontic under the present diagonal discipline; and (vii) we show that each of the three prior hinges (Hyperfactorization, Prime Polarity, Master Constant ι_τ) lifts canonically into $\mathbb{D}$, exhibiting the hinge bundle as four coordinated theorems about a single algebraic object.

Keywords profinite ring, Chinese remainder theorem, split-complex numbers, idempotents, Teichmüller lifts, finite witness predicates, countable foundations, canonical uniqueness theorem, four-atom dictionary, hyperbolic number plane, elliptic exclusion, Panta Rhei hinge paper

MSC 2020 Mathematics Subject Classification: 11A41, 11A07, 03F65, 18A15

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1. INTRODUCTION AND STATEMENT OF RESULTS

1.1 Position in the hinge-paper bundle

This paper is **Hinge 4** of the eight-paper Panta Rhei foundational bundle accompanying the 2nd Edition of the series [?, ?, ?]. The bundle consists of seven technical hinges (H1–H7) plus a foundational-anchor paper (H8); in the recommended reading order they are:

Hinge 1: *Hyperfactorization* [?] — unique tower-atom decomposition $X = (A \uparrow\uparrow C)^B \cdot D$ in ZFC and Category τ, with an Isomorphism Theorem; supplies the ABCD coordinate functions.

Hinge 2: *Prime Polarity* [?] — classifies rational primes into B/C channels via the Legendre symbol $(2/p)$, proved pointwise equivalent to a τ-internal CRT-idempotent-plus-split-complex classifier ($\text{Label}_\infty \equiv \text{Pol}$).

Hinge 3: *Master Constant ι_τ* [?] — structural derivation of $\iota_\tau = 2/(\pi + e) \approx 0.341304$ as the canonical scalar readout of the unique σ-fixed crossing-point ω-germ on the lemniscate, with the split-complex idempotent lift $\tilde{\chi}$ of Prime Polarity's character χ.

Hinge 4: *The Split-Complex Boundary Algebra $\mathbb{D}$ (this paper)* — establishes $\mathbb{D}$ as the unique commutative $\mathcal{R}_\partial$-algebra satisfying four τ-kernel structural constraints; rules out the elliptic $\mathbb{Z}[i]$ alternative; derives the four-atom spectral dictionary.

Hinge 5: *τ-Holomorphy on the Boundary Algebra* [?] — τ-holomorphy as ontological primary; earned categorical machine; wave-equation Cauchy–Riemann; pre-Yoneda collapse.

Hinge 6: *The τ-Topos and Its Four-Valued Internal Logic* [?] — the topos $\mathbf{Cat}_\tau$ with subobject classifier $\Omega_\tau = B_\sigma(\mathbb{D})$ and paraconsistent Belnap–Dunn internal logic.

Hinge 7: *Address Resolution, Not Calculation* [?] — NF confluence (Church–Rosser), genealogical DAG, Cayley word metric, ontic ultrametric.

Hinge 8: *The τ-Kernel as Foundational Architecture* [?] — foundational-anchor paper (also readable as an entry point): ontic identity invariance, diagonal–linear correspondence, $*$-autonomous placement.

Each paper is standalone-readable. Hinge 8 may be read first (as an entry) or last (as a capstone); the other seven may be read in the dependency order above. The present paper (Hinge 4) provides the algebraic skeleton on which Hinges 5–7 rest, and which Hinge 8 audits at the kernel-axioms level.

1.2 Boundary-first scalars

The coherence discipline of the Panta Rhei 2nd-Edition series [?, ?, ?], and of the companion hinge papers [?, ?, ?], forbids local infinity and unearned diagonals. Scalar computation therefore cannot be grounded in uncountable continua. Instead, it is earned *boundary-first*: finite witnesses $\to$ inverse limits $\to$ characters. This paper assembles the resulting scalar layer as a single algebraic object — the split-complex boundary algebra $\mathbb{D}$ — and proves that it is the *unique* such object compatible with the τ-kernel constraints inherited from the prior hinges.

1.3 Main contributions

- A canonical primorial modulus ladder (M_k) and a countable profinite inverse limit ring $\mathcal{R}_\partial$ (Theorem ??).
- A constructive Chinese remainder theorem (CRT) as a theorem of the witness calculus (Theorem ??).
- Teichmüller-style lifts as primitive recursive normalisers compatible with the inverse system (Theorem ??).
- Boundary characters as stabilised ω-tails, yielding a canonical split-complex scalar algebra $\mathbb{D}$ with idempotents $e^\pm$ (Theorem ??).
- A unique crossing mediator ι_τ characterised by balance across the two lobes (Theorem ??).
- **(New in Hinge 4.)** A *canonical uniqueness theorem*: $\mathbb{D}$ is the unique commutative $\mathcal{R}_\partial$-algebra satisfying four explicit structural constraints (binary dimensionality, commutativity, two nontrivial orthogonal idempotents, canonical polarity-swap involution); see Theorem ??.
- **(New in Hinge 4.)** An *elliptic complex exclusion theorem* ruling out $\mathbb{Z}[i]/(i^2+1)$ as a host algebra: in the elliptic setting no nontrivial idempotents exist, hence no B/C decomposition is available (Theorem ??).
- **(New in Hinge 4.)** A *four-atom spectral dictionary* establishing the canonical bijection $\{0, e_+, e_-, 1\} \leftrightarrow \{\alpha\text{-null}, \gamma, \eta, \alpha\text{-total}\}$ with $\iota_\tau \leftrightarrow \omega$, and the explicit 4+1 force-mapping correspondence (Theorem ??).
- **(New in Hinge 4.)** A *hinge integration theorem* showing that all three prior hinges' central objects lift canonically into $\mathbb{D}$ (Theorem ??).

1.4 Main theorems (summary)

Theorem 1.1 (Countable profinite boundary ring **[Established]**)**.** *There is a canonically defined ring $\mathcal{R}_\partial$ presented as an inverse limit $\mathcal{R}_\partial \cong \varprojlim_k \mathbb{Z}/M_k\mathbb{Z}$ over the primorial ladder (M_k), where elements and compatibility are witnessed by finite codes. Moreover, $\mathcal{R}_\partial$ is countable under the present discipline.*

Theorem 1.2 (Constructive CRT **[Established]**)**.** *For each stage k, the canonical map $\mathbb{Z}/M_k\mathbb{Z} \to \prod_{n\le k} \mathbb{Z}/p_n\mathbb{Z}$ is constructively invertible, and these inverses are coherent across the ladder.*

Theorem 1.3 (Teichmüller-style lift **[Established]**)**.** *There exists a canonical primitive-recursive lift operator* Lift *compatible with the inverse system, such that boundary characters factor through* Lift.

Theorem 1.4 (Split-complex scalar algebra **[τ-Effective]**). *Boundary characters induce a canonical split-complex scalar algebra $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2 - 1)$ with orthogonal idempotents $e_+ = \frac{1}{2}(1 + j)$, $e_- = \frac{1}{2}(1 - j)$ corresponding to the bipolar boundary structure derived from prime polarity (Hinge 2 [**?**]; Book I, I.T05).*

Theorem 1.5 (Crossing mediator **[τ-Effective]**). *There exists a unique crossing mediator $\iota_\tau \in \mathbb{D}$ characterised as the unique stabilised balanced element at the junction of the two boundary lobes, with scalar readout $\iota_\tau = 2/(\pi + e) \approx 0.341304$ (Hinge 3 [**?**]).*

Theorem 1.6 (Canonical uniqueness of $\mathbb{D}$ **[τ-Effective]**). *Working over the dyadic localisation $\mathcal{R}'_\partial := \mathcal{R}_\partial[1/2]$ (aligned with Hinge 2's $\chi(2) = 0$ ramification), any commutative $\mathcal{R}'_\partial$-algebra satisfying the four τ-kernel structural constraints — (C1) binary rank (free of rank 2 as $\mathcal{R}'_\partial$-module), (C2) commutativity, (C3) a pair of nontrivial orthogonal idempotents inherited from the B/C prime bipartition, and (C4) a canonical polarity-swap involution σ exchanging the two idempotents — is canonically isomorphic to $\mathbb{D} = \mathcal{R}'_\partial[j]/(j^2 - 1)$, with the isomorphism sending the idempotent pair to (e_+, e_-) and the involution to $j \mapsto -j$ (Theorem **??** below).*

Theorem 1.7 (Elliptic complex exclusion **[Established]**). *Over $\mathcal{R}_\partial[1/2]$, the commutative algebra $A = \mathcal{R}_\partial[i]/(i^2 + 1)$ admits no σ-equivariant pair of nontrivial orthogonal idempotents — equivalently, no pair $e_\pm \in A$ with $e_+ + e_- = 1$, $e_+e_- = 0$, $e_\pm \notin \{0, 1\}$, and $\sigma_{\text{ell}}(e_+) = e_-$ for the canonical involution $\sigma_{\text{ell}}(i) = -i$. Consequently the elliptic Gaussian structure is structurally incompatible with the τ-kernel's B/C bipartition, and the canonical involution $i \mapsto -i$ is a Galois rotation, not a polarity swap (Theorem **??** below). CRT-local idempotents at primes $p \equiv 1 \pmod 4$ do exist in A, but they are not σ-equivariant in the globally-required sense.*

Theorem 1.8 (Four-atom spectral dictionary **[τ-Effective]**). *Within $\mathbb{D}$, the canonical σ-equivariant Boolean sublattice $B_\sigma(\mathbb{D})$ of idempotents — the smallest Boolean sub-$*$-algebra of $\operatorname{Idem}(\mathbb{D})$ closed under σ and containing the canonical pair $\{e_+, e_-\}$ — has exactly four elements $\{0, e_+, e_-, 1\}$ and is in canonical bijection with four channel-eigenstates of the lemniscate boundary: $0 \leftrightarrow \alpha$-null, $e_+ \leftrightarrow \gamma$ (EM), $e_- \leftrightarrow \eta$ (strong), $1 \leftrightarrow \alpha$-total (gravity). The unique non-idempotent σ-fixed scalar ι_τ corresponds to the ω-generator (Higgs mediator), and the fifth generator π (weak) lives at the base-refinement level outside $\mathbb{D}$, giving the $4{+}1$ sector structure of the Panta Rhei physics stratum (Theorem **??** below). The full idempotent set $\operatorname{Idem}(\mathbb{D})$ is much larger (profinitely many idempotents from the CRT decomposition of $\mathcal{R}_\partial$); the four atoms of $B_\sigma(\mathbb{D})$ are the structural ones forced by $j^2 = +1$ itself, not by the prime-by-prime decomposition of $\mathcal{R}_\partial$.*

Theorem 1.9 (Hinge integration in $\mathbb{D}$ **[τ-Effective]**). *All three prior hinges' central objects lift canonically into $\mathbb{D}$: the Hyperfactorization ABCD coordinates extend to $\mathbb{D}$-valued functions by idempotent-componentwise action; the Prime Polarity character χ lifts to the split-complex character $\tilde{\chi} : (\mathbb{N}, \times) \to (\mathbb{D}, +)$; and the Master Constant ι_τ is the ω-atom of the dictionary. Evaluating Hinge 1 coordinates on Hinge-2-classified primes along the primorial ladder recovers ι_τ in the refinement limit (Theorem **??** below).*

1.5 Relationship to orthodox profinite/adic structures

We do not assume full $\widehat{\mathbb{Z}}$ or uncountable topological bases as primitives. The inverse limit is earned by finitary compatibility predicates and stabilization. This is the constructive shadow of $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$, earned from the primorial system and witness predicates rather than imposed.

2. STANDING SETUP: MODULI LADDERS AND FINITE WITNESS PREDICATES

Definition 2.1 (Primorial modulus ladder). *Let $(p_n)_{n \geq 1}$ be the prime sequence. Define the canonical primorial ladder*

$$M_k := \prod_{n=1}^{k} p_n.$$

Remark 2.2. The primorial ladder is canonical: it is the minimal global multiplicative fold over the prime orbit. Any per-prime ladder yields local fibers; the primorial yields the global refinement ladder.

Definition 2.3 (Reduction maps). *For $k \le \ell$, define $\pi_{\ell\to k} \colon \mathbb{Z}/M_\ell\mathbb{Z} \to \mathbb{Z}/M_k\mathbb{Z}$ by reduction modulo M_k.*

Definition 2.4 (Finitary compatibility predicate). *A (candidate) tower is a family $(x_k)_k$ with $x_k \in \mathbb{Z}/M_k\mathbb{Z}$. It is* compatible *if for all $k \le \ell$, $\pi_{\ell\to k}(x_\ell) = x_k$, with equality verified by finite witnesses. In Lean, this is implemented as a decidable predicate with an explicit witness budget bound.*

3. THE COUNTABLE PROFINITE BOUNDARY RING

Definition 3.1 (Boundary ring). *Define*

$$\mathcal{R}_\partial := \varprojlim_k \mathbb{Z}/M_k\mathbb{Z}$$

as the type of compatible towers modulo tail-stable extensional equality.

Lemma 3.2 (Ring operations are definitional). *Addition and multiplication on $\mathcal{R}_\partial$ are induced stagewise from $\mathbb{Z}/M_k\mathbb{Z}$ and preserve compatibility.*

Lean-grade sketch. Stagewise operations commute with reductions; compatibility witnesses compose. □

Lemma 3.3 (Countability). *$\mathcal{R}_\partial$ is countable: every element is represented by a finite code producing a compatible tower and witnesses.*

Lean-grade sketch. Codes are countable; each element corresponds to at least one code. □

4. CONSTRUCTIVE CRT AT FINITE STAGES AND IN THE LIMIT

Theorem 4.1 (Finite-stage CRT). *For each k, the map*

$$\phi_k \colon \mathbb{Z}/M_k\mathbb{Z} \to \prod_{n=1}^{k} \mathbb{Z}/p_n\mathbb{Z}$$

is an isomorphism with an explicitly definable inverse ψ_k.

Lean-grade sketch. Standard CRT specialized to pairwise coprime primes; build idempotents E_n modulo M_k. □

Theorem 4.2 (CRT coherence across the ladder). *The family $(\phi_k, \psi_k)_k$ is coherent with respect to the reduction maps $\pi_{\ell\to k}$.*

Corollary 4.3 (Profinite CRT (constructive)). *There is a constructive decomposition of boundary towers into coherent local factors, yielding the internal analogue of $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$.*

5. TEICHMÜLLER-STYLE LIFTS AS PRIMITIVE RECURSIVE NORMALIZERS

Definition 5.1 (Lift operator (stagewise)). *A Teichmüller-style lift is a compatible family of maps*

$$\mathsf{Lift}_k \colon \mathbb{Z}/M_k\mathbb{Z} \to \mathcal{R}_\partial$$

such that $\pi_{\infty\to k}(\mathsf{Lift}_k(x)) = x$ and Lift is stable under refinement.

Theorem 5.2 (Existence of canonical lift). *There exists a canonical primitive-recursive lift operator Lift definable from the primorial ladder and CRT idempotents.*

Lean-grade sketch. Define Lift by choosing canonical representatives at each stage (via ψ_k), then stabilize in the limit. □

Lemma 5.3 (Character factorization through Lift). *Every stabilized boundary character factors through Lift, i.e. is determined by its lifted tower data.*

6. BOUNDARY CHARACTERS AND THE SPLIT-COMPLEX SCALAR ALGEBRA

Definition 6.1 (Boundary characters). *A boundary character is a tail-stable evaluation functional on $\mathcal{R}_\partial$, implemented as a stabilized ω-tail (finite-stage dependence, infinite stabilization). Denote the character type by* Char_τ.

Definition 6.2 (Split-complex algebra). *Define the split-complex algebra over $\mathcal{R}_\partial$ by*

$$\mathbb{D} := \{a + bj \mid a, b \in \mathcal{R}_\partial,\ j^2 = +1\}.$$

Equivalently, $\mathbb{D} \cong \mathcal{R}_\partial \times \mathcal{R}_\partial$ via orthogonal idempotents e_+, e_-.

Remark 6.3. We do *not* import elliptic complex numbers as primitives. The earned scalar algebra is hyperbolic (split-complex).

Theorem 6.4 (Canonical idempotents **[τ-Effective]**). *There exist canonical orthogonal idempotents $e_+, e_- \in \mathbb{D}$ with $e_+^2 = e_+$, $e_-^2 = e_-$, $e_+e_- = 0$, and $e_+ + e_- = 1$, induced by the bipolar boundary structure.*

Lean-grade sketch. Construct via the bipolar spectrum projector determined by prime polarity (Hinge 2, [?]; Book I [?], Theorem I.T05). ☐

Theorem 6.5 (Canonical idempotent splitting). *The boundary character algebra carries a canonical decomposition:*

$$\mathsf{Char}_\tau \ \cong\ \mathsf{Char}_B \times \mathsf{Char}_C,$$

with corresponding idempotent projectors e_+, e_- and scalar algebra $\mathbb{D}$.

7. THE CROSSING MEDIATOR ι_τ

Definition 7.1 (Crossing mediator). *The crossing mediator $\iota_\tau \in \mathbb{D}$ is the unique stabilized balanced element characterized by:*

(a) non-degeneracy: $e_+\iota_\tau \neq 0$ and $e_-\iota_\tau \neq 0$,
(b) balance: ι_τ is a fixed point or minimizer of the canonical lobe-balance functional,
(c) stability: ι_τ is tail-stable (an ω-germ).

Theorem 7.2 (Existence and uniqueness of ι_τ **[τ-Effective]**). *There exists a unique ι_τ satisfying Definition* **??**.

Lean-grade sketch. Existence: Construct the canonical balanced germ by alternating refinement and applying stabilization.

Uniqueness: Two distinct balanced elements would define two distinct junctions, contradicting the single junction of the wedge $\mathbb{L} = S^1 \vee S^1$. ☐

Remark 7.3. The value $\iota_\tau = 2/(\pi + e) \approx 0.3413\ldots$ is the unique balanced element; it emerges from the structural balance condition with no free parameters.

8. THE UNIQUENESS THEOREM

The preceding sections have established a chain of structural observations: the boundary algebra $\mathcal{R}_\partial$ admits a canonical involution σ (§**??**), the Chinese Remainder decomposition forces a pair of orthogonal idempotents e_+, e_- with $e_+ + e_- = 1$ (§**??**, Theorem **??**), and the split-complex ring $\mathbb{Z}[j]/(j^2 - 1)$ presents itself as the smallest commutative ring carrying both pieces of data (Definition **??**). What remains is to prove that this is no coincidence: the split-complex structure is *forced* by four minimal structural requirements, and no alternative (in particular, no elliptic $j^2 = -1$) can satisfy them.

This is the content of the present section. We isolate the four constraints in §**??**, state the uniqueness theorem in §**??**, give the full proof in §**??**, and extract corollaries in §**??**. The argument is elementary ring theory in the style of [**?**, Ch. 2], but its significance for the wider Hinge programme is substantial: it is the structural lemma behind the polarity rigidity results of [**?**] and the spectral uniqueness of [**?**].

Remark 8.1 (Dyadic localization of the boundary ring). Throughout this section we work in the *dyadic localization*

$$\mathcal{R}'_\partial \;:=\; \mathcal{R}_\partial[\tfrac{1}{2}] \;=\; \mathcal{R}_\partial \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}],$$

of the countable profinite boundary ring $\mathcal{R}_\partial$ of Theorem **??**, so that the idempotents $e_+ = (1+j)/2$ and $e_- = (1-j)/2$ are integral. This passage is forced upon us: the Hinge 2 character $\chi\colon (\mathbb{Z}/D)^\times \to \{\pm 1\}$ constructed in [**?**] satisfies $\chi(2) = 0$ (the dyadic prime is ramified in the polarity decomposition, see [**?**, Ch. 5]), so 2 is excluded from the multiplicative set of "addressable" primes in the sense of [**?**, §4]. Inverting 2 on the boundary algebra is therefore not a loss of information but an alignment with the natural site of the theory.

All statements in this section should be read over $\mathcal{R}'_\partial$: the base ring for the uniqueness theorem is $\mathcal{R}'_\partial$, not $\mathbb{Z}[\frac{1}{2}]$. The relationship is that $\mathbb{Z}[\frac{1}{2}] \hookrightarrow \mathcal{R}'_\partial$ is the dyadic localization of the prime subring of $\mathcal{R}_\partial$, and all proofs below use only the ring-theoretic fact that 2 is a unit in $\mathcal{R}'_\partial$; no property specific to $\mathbb{Z}[\frac{1}{2}]$ is invoked. Consequently the uniqueness conclusion $A \cong \mathcal{R}'_\partial[j]/(j^2-1)$ holds for any $\mathcal{R}'_\partial$-algebra A satisfying the four structural constraints below, in agreement with Theorem **??**.

8.1 The four structural constraints

We now isolate, in purely ring-theoretic terms, the minimal structural requirements that the boundary algebra must satisfy. Each of the four constraints below has been motivated at length in the preceding sections; our task here is to collect them into a rigid axiomatic package.

Definition 8.2 (Boundary-algebra axioms). *Let A be a commutative $\mathcal{R}'_\partial$-algebra, equipped with an $\mathcal{R}'_\partial$-algebra involution $\sigma\colon A \to A$ (that is, an $\mathcal{R}'_\partial$-algebra automorphism with $\sigma^2 = \mathrm{id}_A$). We say that the pair (A, σ) is a* boundary-algebra datum *if the following four conditions hold:*

- (C1) ***Binary rank.*** *A is free of rank 2 as an $\mathcal{R}'_\partial$-module;*
- (C2) ***Commutativity.*** *A is commutative (as a ring);*
- (C3) ***Idempotent pair.*** *A contains a pair of nontrivial orthogonal idempotents $e_+, e_- \in A$ with $e_+ + e_- = 1_A$, $e_+e_- = 0$, and $e_\pm \notin \{0, 1_A\}$;*
- (C4) ***Involution swap.*** *The involution σ swaps the idempotents: $\sigma(e_+) = e_-$ (equivalently $\sigma(e_-) = e_+$).*

Remark 8.3 (Provenance of the four constraints).

- (C1) is forced by the wedge-boundary geometry $\mathbb{L} = S^1 \vee S^1$: a single wedge point joining two circles has cohomology of rank 2 in degree 1, giving the boundary functor a two-dimensional target ([**?**, Ch. 3]).
- (C2) is the commutativity of the boundary operator established in Theorem **??**: multiplication on $\mathcal{O}(\tau^3)$ is commutative, so its boundary image must be too.
- (C3) is the Chinese Remainder decomposition (§**??**): the two wedge branches project onto orthogonal summands.
- (C4) is the σ-involution axiom of §**??**: the wedge map that exchanges the two circles must exchange the corresponding idempotents ([**?**, Ch. 5]).

Lemma 8.4 (Rank splitting along the idempotent pair). *Let (A, σ) satisfy* (C1)–(C4)*. Then e_+A and e_-A are each free $\mathcal{R}'_\partial$-modules of rank exactly 1, and $A = e_+A \oplus e_-A$ as $\mathcal{R}'_\partial$-modules.*

Proof. Orthogonality and $e_+ + e_- = 1_A$ give the internal direct-sum decomposition $A = e_+A \oplus e_-A$ as $\mathcal{R}'_\partial$-modules: every $a \in A$ equals $e_+a + e_-a$, and $e_+A \cap e_-A = \{0\}$ because $e_+e_- = 0$. Each summand e_+A is a direct summand of the rank-2 free $\mathcal{R}'_\partial$-module A, hence finitely generated and projective; so is e_-A. Since $\mathcal{R}'_\partial$ is a commutative Noetherian ring in which every finitely generated projective module has a well-defined rank additive on direct sums, we have $\mathrm{rank}_{\mathcal{R}'_\partial}(e_+A) + \mathrm{rank}_{\mathcal{R}'_\partial}(e_-A) = 2$. By (C3) each e_+A, e_-A is nonzero (else e_+ or e_- would lie in $\{0, 1_A\}$), so each rank is at least 1; the sum is 2, hence each is exactly 1. A rank-1 projective module over a local ring (and hence after passing to any stalk) is free, and e_+A is free on the generator e_+ because $(r \mapsto re_+)$ is injective (if $re_+ = 0$ then multiplying by e_+ on the right and using $e_+^2 = e_+$ gives $re_+ = 0$, and inspecting the rank-2 free structure shows $r = 0$). Similarly for e_-A. □

Lemma 8.5 (Idempotent normalization). *Let (A, σ) satisfy (C1)–(C4) and pick an idempotent decomposition $1 = e_+ + e_-$ with $\sigma(e_+) = e_-$. Set*

$$j := e_+ - e_- \in A.$$

Then:

(i) $j^2 = 1_A$;

(ii) $\sigma(j) = -j$;

(iii) $\{1_A, j\}$ *is a free $\mathcal{R}'_\partial$-basis of A.*

Proof. *(i)* Using orthogonality and the idempotent identities:

$$j^2 = (e_+ - e_-)^2 = e_+^2 - 2e_+e_- + e_-^2 = e_+ - 0 + e_- = 1_A.$$

(ii) By (C4) and $\mathcal{R}'_\partial$-linearity of σ:

$$\sigma(j) = \sigma(e_+) - \sigma(e_-) = e_- - e_+ = -j.$$

(iii) Since 2 is a unit in $\mathcal{R}'_\partial$, the $\mathcal{R}'_\partial$-linear change of basis

$$(1_A, j) \longleftrightarrow (e_+, e_-), \qquad e_+ = \tfrac{1}{2}(1_A + j), \quad e_- = \tfrac{1}{2}(1_A - j), \quad 1_A = e_+ + e_-, \quad j = e_+ - e_-,$$

is invertible over $\mathcal{R}'_\partial$. By Lemma ??, $\{e_+, e_-\}$ is a free $\mathcal{R}'_\partial$-basis of $A = e_+A \oplus e_-A$ (each summand being free of rank 1); the change of basis above therefore carries this to the assertion that $\{1_A, j\}$ is also a free $\mathcal{R}'_\partial$-basis of A. □

Remark 8.6 (The element j as a structural invariant). Lemma ?? shows that the element $j \in A$ is canonically determined *up to sign* by the axioms (C1)–(C4): swapping $(e_+, e_-) \to (e_-, e_+)$ sends $j \to -j$, and this is the only ambiguity. This sign ambiguity is the algebraic trace of the wedge-orientation choice discussed in Definition ??. We fix a sign once and for all.

8.2 Statement of the uniqueness theorem

We are now ready to state the main structural result of this paper.

Theorem 8.7 (Uniqueness of the split-complex boundary algebra). *Let (A, σ) be a boundary-algebra datum in the sense of Definition ??. Then there is a unique $\mathcal{R}'_\partial$-algebra isomorphism*

$$\Phi_{\text{unique}} : A \xrightarrow{\sim} \mathcal{R}'_\partial[j]/(j^2 - 1) = \mathcal{R}_\partial[\tfrac{1}{2}][j]/(j^2 - 1)$$

that sends $e_+ \mapsto \frac{1}{2}(1 + j)$ and intertwines σ with the involution $j \mapsto -j$.

In particular: any commutative $\mathcal{R}'_\partial$-algebra of free rank 2 that admits a nontrivial orthogonal idempotent pair and an $\mathcal{R}'_\partial$-algebra involution exchanging them is canonically isomorphic to the split-complex boundary algebra $\mathcal{R}'_\partial[j]/(j^2-1)$, with no freedom in the isomorphism. This is the $\mathcal{R}'_\partial$-algebra refinement of Theorem ??; via the base-change functor $\mathcal{R}_\partial[1/2] \otimes_{\mathcal{R}_\partial} (-) : \mathcal{R}_\partial\text{-}Alg \to \mathcal{R}'_\partial\text{-}Alg$, it recovers the stated uniqueness over $\mathcal{R}_\partial$-algebras in which 2 is invertible.

Remark 8.8 (Why "uniqueness" is strong). Theorem ?? rules out three a priori plausible alternatives:

(a) the *elliptic* alternative $\mathcal{R}'_\partial[i]/(i^2 + 1)$ (Gaussian extension of $\mathcal{R}'_\partial$);

(b) the *dual-number* alternative $\mathcal{R}'_\partial[\varepsilon]/(\varepsilon^2)$ (which would be non-semisimple and lacks the idempotent pair);

(c) any "twisted" variant $\mathcal{R}'_\partial[j]/(j^2 - c)$ with c a unit other than 1.

We address (a) in detail in Corollary ?? below; (b) and (c) are precluded by the idempotent and involution constraints respectively.

8.3 Proof of the uniqueness theorem

The proof proceeds in four steps: basis expansion, minimal polynomial, application of the involution swap, and the $j^2 = +1$ conclusion. Each step is an elementary computation, but together they yield the rigidity statement. We first record a one-line lemma extracting the scalar-fixing property of σ that is used in Step 2.

Lemma 8.9 (The involution fixes scalars). *Let A be a commutative $\mathcal{R}'_\partial$-algebra and let $\sigma\colon A \to A$ be an $\mathcal{R}'_\partial$-algebra automorphism of order 2. Then $\sigma(r \cdot 1_A) = r \cdot 1_A$ for every $r \in \mathcal{R}'_\partial$; that is, the scalar subring $\mathcal{R}'_\partial \cdot 1_A \subseteq A$ is contained in the σ-fixed subring A^σ.*

Proof. By hypothesis σ is $\mathcal{R}'_\partial$-linear: for all $r \in \mathcal{R}'_\partial$ and $a \in A$, $\sigma(r \cdot a) = r \cdot \sigma(a)$. Taking $a = 1_A$ and using $\sigma(1_A) = 1_A$ (every ring automorphism fixes the unit), we get $\sigma(r \cdot 1_A) = r \cdot 1_A$, as claimed. ☐

Proof of Theorem **??**. Let (A, σ) satisfy (C1)–(C4). By Lemma **??**, the element $j = e_+ - e_-$ satisfies $j^2 = 1$, $\sigma(j) = -j$, and $\{1, j\}$ is a free $\mathcal{R}'_\partial$-basis of A. Define

$$\Phi_{\text{unique}}\colon \mathcal{R}'_\partial[T] \to A, \qquad T \mapsto j.$$

Since $j^2 - 1 = 0$ in A, this factors through the quotient

$$\bar{\Phi}_{\text{unique}}\colon \mathcal{R}'_\partial[T]/(T^2 - 1) \twoheadrightarrow A.$$

$\mathcal{R}'_\partial$-linear independence of $\{1, j\}$ makes $\bar{\Phi}_{\text{unique}}$ injective, hence an isomorphism of $\mathcal{R}'_\partial$-algebras. It remains to show that $\bar{\Phi}_{\text{unique}}$ is the *unique* such isomorphism with the stated intertwining property, and that no other polynomial relation $T^2 - c = 0$ with $c \neq 1$ could replace $T^2 - 1$.

Step 1 (Basis expansion). By (C1) and Lemma **??**(iii), every element $a \in A$ can be written uniquely as $a = \alpha + \beta j$ with $\alpha, \beta \in \mathcal{R}'_\partial$.

Step 2 (Minimal polynomial of j). The element $j \in A$ satisfies a monic quadratic relation over $\mathcal{R}'_\partial$, because $\{1, j, j^2\}$ must be $\mathcal{R}'_\partial$-linearly dependent by (C1). Write

$$j^2 \;=\; \alpha + \beta j \qquad (\alpha, \beta \in \mathcal{R}'_\partial).$$

Applying σ and using $\sigma(j) = -j$ from Lemma **??**(ii) together with the fact that σ fixes the scalar subring $\mathcal{R}'_\partial \cdot 1_A$ pointwise (Lemma **??**):

$$(-j)^2 \;=\; \alpha + \beta(-j) \qquad \Longleftrightarrow \qquad j^2 \;=\; \alpha - \beta j.$$

Comparing with $j^2 = \alpha + \beta j$ forces $\beta = -\beta$, i.e. $2\beta = 0$ in $\mathcal{R}'_\partial$. Since 2 is a unit in $\mathcal{R}'_\partial$ (Remark **??**), we conclude $\beta = 0$. Thus $j^2 = \alpha$ for some $\alpha \in \mathcal{R}'_\partial$.

Step 3 (Determination of α via idempotents). Recall that (C3) provides the idempotent e_+ with $e_+^2 = e_+$ and $e_+ \neq 0, 1$. From $e_+ = \frac{1}{2}(1 + j)$ (which follows from the basis expansion and the normalization $j = e_+ - e_-$, $1 = e_+ + e_-$):

$$e_+^2 \;=\; \tfrac{1}{4}(1 + j)^2 \;=\; \tfrac{1}{4}(1 + 2j + j^2) \;=\; \tfrac{1}{4}(1 + 2j + \alpha).$$

Setting this equal to $e_+ = \frac{1}{2}(1 + j)$ and multiplying through by 4 (legitimate because $2 \in \mathcal{R}'^{\times}_\partial$, so $4 \in \mathcal{R}'^{\times}_\partial$):

$$1 + 2j + \alpha \;=\; 2 + 2j, \qquad \Longrightarrow \qquad \alpha \;=\; 1.$$

Equivalently, the orthogonality condition $e_+e_- = 0$ reads $\frac{1}{4}(1 - j)(1 + j) = \frac{1}{4}(1 - j^2) = 0$, and since 4 is a unit in $\mathcal{R}'_\partial$ this forces $j^2 = 1_A$ directly. Either way, $\alpha = 1$, and the minimal polynomial of j over $\mathcal{R}'_\partial$ is $T^2 - 1$, not $T^2 + 1$ or any other quadratic.

Step 4 (Uniqueness of Φ_{unique}). Suppose $\Phi'\colon \mathcal{R}'_\partial[T]/(T^2 - 1) \to A$ is another $\mathcal{R}'_\partial$-algebra isomorphism sending $T \mapsto j'$ where $j' \in A$ and Φ' sends $\frac{1}{2}(1 + T) \mapsto e_+$. Then $\frac{1}{2}(1 + j') = e_+$, whence $j' = 2e_+ - 1 = e_+ - e_- = j$. Thus $\Phi' = \bar{\Phi}_{\text{unique}}$, and uniqueness is established.

The $j^2 = +1$ conclusion. Steps 2–3 together give the decisive structural fact: the involution swap (C4), together with the idempotent structure (C3) and the invertibility of 2 in $\mathcal{R}'_\partial$, forces $j^2 = +1$ (the *hyperbolic* or *split* case), and rules out $j^2 = -1$ (the elliptic case). This is the algebraic shadow of the wedge-point geometry: the two circles of $\mathbb{L} = S^1 \vee S^1$ meet at a *real* fixed point of the involution, not at a purely imaginary one, and it is this reality of the wedge point that propagates to the $+1$ sign in the ring relation. The geometric discussion in [?, Ch. 4] makes this picture precise. □

Remark 8.10 (On the role of the $\mathcal{R}'_\partial$-rank-2 condition)**.** The rank-2 condition (C1) plays two distinct roles in the proof: it gives the $\mathcal{R}'_\partial$-linear dependence of $\{1, j, j^2\}$ used in Step 2, and it makes the putative algebra "as small as possible," so the proof establishes a minimality statement. If we relaxed (C1) to allow higher $\mathcal{R}'_\partial$-rank, the involution-swap argument would still force j^2 to be a scalar in $\mathcal{R}'_\partial \cdot 1_A$, but one would have to exclude extraneous idempotent decompositions.

8.4 Corollaries

We now extract four structural consequences of Theorem ??. The first three are immediate; the fourth is the comparison with the elliptic alternative.

Corollary 8.11 (Uniqueness of σ)**.** *Let (A, σ) satisfy* (C1)–(C4)*. Then σ is the* unique *nontrivial ring involution on A that swaps a pair of nontrivial orthogonal idempotents. Any two such involutions agree.*

Proof. By Theorem ??, $(A, \sigma) \cong (\mathcal{R}'_\partial[j]/(j^2 - 1), j \mapsto -j)$ via a unique $\mathcal{R}'_\partial$-algebra isomorphism. Suppose σ' is another such involution. Then $\sigma'(j)$ must be an element squaring to 1, $\mathcal{R}'_\partial$-linearly independent from 1, and *not* equal to j (else σ' would be trivial on the basis and hence identity). The only such element is $-j$, so $\sigma'(j) = -j = \sigma(j)$, whence $\sigma' = \sigma$. □

Corollary 8.12 (Uniqueness of the idempotent structure)**.** *Under the hypotheses of Theorem ??, the pair of nontrivial orthogonal idempotents $\{e_+, e_-\}$ is uniquely determined (as an unordered pair) by the ring structure of A alone; the choice of ordering is an σ-equivariant sign convention.*

Proof. The nontrivial idempotents of $\mathcal{R}'_\partial[j]/(j^2 - 1)$ are exactly the two elements $\frac{1}{2}(1 \pm j)$: any element $a = \alpha + \beta j$ with $\alpha, \beta \in \mathcal{R}'_\partial$ satisfies $a^2 = a$ iff $\alpha^2 + \beta^2 = \alpha$ and $2\alpha\beta = \beta$, i.e. $\beta(2\alpha - 1) = 0$ and $\alpha(\alpha - 1) = -\beta^2$. Since $\mathcal{R}'_\partial$ has no idempotents other than 0, 1 outside the CRT components tracked by Theorem ??, either $\beta = 0$ (yielding $\alpha \in \{0, 1\}$, the trivial idempotents) or $2\alpha - 1 = 0$, whence $\alpha = 1/2$ and $\beta^2 = 1/4$, i.e. $\beta = \pm 1/2$. This matches the Chinese Remainder decomposition of Theorem ?? and its profinite version Corollary ??. □

Corollary 8.13 (Rigidity of the ι_τ scalar)**.** *The constant $\iota_\tau = 2/(\pi + e)$ enters $A \otimes_{\mathbb{Z}} \mathbb{R}$ as the unique scalar consistent with the uniqueness theorem and the normalization of Theorem ??. In particular, no other real scalar can play the role of ι_τ in the boundary algebra presentation.* **[τ-Effective]**

Proof. Combine Theorem ?? with the normalization argument of [?, §5]: the scalar ι_τ is fixed by a spectral condition that, once (A, σ) is identified with the split-complex ring $\mathcal{R}'_\partial[j]/(j^2 - 1)$, admits a unique solution in $(0, 1) \subset \mathbb{R}$. □

Corollary 8.14 (No elliptic alternative)**.** *The elliptic ring $E := \mathcal{R}'_\partial[i]/(i^2 + 1)$ does* not *admit a boundary-algebra datum structure satisfying* (C1)–(C4)*: there is no $\mathcal{R}'_\partial$-algebra involution σ_E on E swapping a pair of nontrivial orthogonal idempotents.*

Proof. We argue by contradiction. Suppose $\sigma_E \colon E \to E$ is an $\mathcal{R}'_\partial$-algebra involution swapping nontrivial idempotents e_+, e_-. Setting $j := e_+ - e_-$ and running the argument of Steps 2–3 of the proof of Theorem ??, we obtain $j^2 = 1$ in E. But in $E = \mathcal{R}'_\partial[i]/(i^2 + 1)$, any element $a = \alpha + \beta i$ with $\alpha, \beta \in \mathcal{R}'_\partial$ satisfies

$$a^2 = (\alpha^2 - \beta^2) + 2\alpha\beta\, i.$$

Setting $a^2 = 1$ gives $\alpha^2 - \beta^2 = 1$ and $\alpha\beta = 0$ in $\mathcal{R}'_\partial$. Hence either $\beta = 0$, yielding $\alpha^2 = 1$ and $a = \pm 1$ (trivial), or $\alpha = 0$, yielding $-\beta^2 = 1$, which has no solution in $\mathcal{R}'_\partial \hookrightarrow \mathcal{R}_\partial \otimes \mathbb{R}$ (all real completions of $\mathcal{R}'_\partial$ remain totally real). Therefore E contains no nontrivial square root of 1, so the putative j cannot exist, and the idempotent pair $\{e_\pm\}$ cannot be constructed in E. This contradicts the hypothesis.

Geometrically: $\mathcal{R}'_\partial[i]/(i^2 + 1)$ has no zero-divisors at each totally real stalk, hence contains no nontrivial idempotents transverse to the base; the CRT-style decomposition that underlies (C3) is structurally absent. **[Established]** □

Remark 8.15 (Comparison table). The following comparison, adapted from [?, Table 1.1], summarizes the structural difference between the split and elliptic candidates:

Property	Split: $\mathcal{R}'_\partial[j]/(j^2-1)$	Elliptic: $\mathcal{R}'_\partial[i]/(i^2+1)$
Rank over $\mathcal{R}'_\partial$	2	2
Commutative	yes	yes
Zero-divisors	yes (via $e_+e_- = 0$)	no (domain on each stalk)
Nontrivial idempotents	$\frac{1}{2}(1 \pm j)$	none
Swap involution σ	$j \mapsto -j$	$i \mapsto -i$ (complex conj.)
Idempotent swap	yes	**no** (no idempotents to swap)
CRT decomposition	$\mathcal{R}'_\partial \oplus \mathcal{R}'_\partial$	irreducible
Satisfies (C1)–(C4)	**yes**	**no**

Only the split-complex ring satisfies the full axiomatic package; the elliptic alternative fails at (C3).

Corollary 8.16 (The Mayer–Vietoris alignment). *Under the identification of Theorem* **??**, *the Mayer–Vietoris sequence M_k for the wedge $\mathbb{L} = S^1 \vee S^1$ decomposes along the idempotent splitting e_+, e_-, with σ acting as the involution that exchanges the two summands. The match between the geometric Mayer–Vietoris map and the algebraic CRT-decomposition is not coincidental but is forced by the uniqueness theorem.* **[τ-Effective]**

Proof. Immediate from Theorem **??**, Corollary **??**, and the geometric identification in Definition **??**; the detailed unwinding is in [?, §3]. □

Remark 8.17 (Programmatic consequences for the Hinge sequence). Theorem **??** is the structural linchpin of the four-paper Hinge sequence:

- Hinge 1 [?] uses the split structure to realize the wedge as the hyperfact;
- Hinge 2 [?] uses the σ-involution to define prime polarity via the idempotent action;
- Hinge 3 [?] uses the spectral rigidity (Corollary **??**) to derive the ι_τ value;
- Hinge 4 (this paper) completes the package by showing all three uses rest on the same underlying algebra, uniquely determined.

This is the sense in which the boundary algebra is a *hinge*: it is the shared algebraic pivot on which each of the four structural theorems turns. **[Conjectural]**

Remark 8.18 (Limits of the uniqueness statement). Theorem **??** establishes uniqueness of the boundary algebra as a commutative $\mathcal{R}'_\partial$-algebra with involution. It does *not* address:

(i) variations of the base ring beyond dyadic localization (e.g. replacing $\mathcal{R}'_\partial$ with an odd-prime completion $\mathcal{R}_\partial \otimes \mathbb{Z}_p$, p odd, or passing to an integral form where 2 is not inverted);

(ii) non-commutative alternatives (e.g. a rank-2 quaternionic lattice over $\mathcal{R}'_\partial$);

(iii) graded or differential-graded enhancements.

We return to (i) in §**??** below; (ii) and (iii) are deferred to Hinge 5 and beyond. For the present paper, the rigidity result in the commutative setting over $\mathcal{R}'_\partial$ is sufficient to close the Hinge 4 structural loop.

Proposition 8.19 (Summary of §??). *Taken together, Theorem* **??** *and Corollaries* **??**–**??** *establish:*

(a) *The split-complex boundary algebra $\mathcal{R}'_\partial[j]/(j^2-1)$ is* uniquely *characterized, up to unique $\mathcal{R}'_\partial$-algebra isomorphism, by the four axioms* (C1)–(C4)*;*

(b) *The involution σ, the idempotent pair $\{e_+, e_-\}$, and the element j are structurally determined, with sign conventions as the only residual freedom;*

(c) *The elliptic alternative ($j^2 = -1$) is* ruled out*;*

(d) *The split-complex structure $j^2 = +1$ is the unique output of the four-constraint package, and consequently is the unique algebraic model for the wedge-boundary geometry of the τ-framework.*

Proof. Each statement has been established above: (a) is Theorem **??**; (b) combines Lemma **??**, Corollary **??**, and Corollary **??**; (c) is Corollary **??**; (d) is the $j^2 = +1$ conclusion of the proof of Theorem **??**, together with Corollary **??**. □

The uniqueness theorem thus closes the structural investigation begun in §**??**. In §**??** we discuss how the rigidity persists under natural extensions of the base ring; in §**??** we survey consequences for the three earlier Hinge papers.

9. THE ELLIPTIC COMPLEX EXCLUSION THEOREM

9.1 The structural gap: what elliptic complex structure cannot do

The boundary algebra $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2 - 1)$ constructed in Definition **??** is split-complex, not elliptic. Before proving the exclusion theorem, it is worth stating clearly *why* this matters — what the elliptic alternative fails to deliver, not at the level of taste or aesthetics but at the level of algebraic structure that the B/C prime bipartition demands.

The two-dimensional commutative $\mathcal{R}_\partial$-algebras defined by a single quadratic relation $x^2 = c \cdot 1$ fall into three isomorphism classes, distinguished by the sign of c (modulo squares in $\mathcal{R}_\partial$):

(i) $c = -1$: *elliptic complex*, $A \cong \mathcal{R}_\partial[i]/(i^2 + 1)$, generated by a unit i with $i^2 = -1$;

(ii) $c = 0$: *dual numbers* (parabolic), $A \cong \mathcal{R}_\partial[\varepsilon]/(\varepsilon^2)$, with a nilpotent;

(iii) $c = +1$: *split-complex* (hyperbolic), $A \cong \mathcal{R}_\partial[j]/(j^2 - 1)$, with $j^2 = +1$.

Each class carries a canonical involution $x \mapsto \bar{x}$ fixing $\mathcal{R}_\partial \cdot 1$; these are $i \mapsto -i$, $\varepsilon \mapsto -\varepsilon$, and $j \mapsto -j$ respectively. What distinguishes the three is the *idempotent structure*: case (iii) has two non-trivial idempotents e_+, e_-, whereas (i) and (ii) have none. The B/C bipartition inherited from prime polarity (Hinge 2, [**?**]) is precisely a commuting pair of orthogonal projectors summing to 1; in the three quadratic algebras, only (iii) hosts such a decomposition as an algebraic feature rather than as externally imposed structure. The theorem below makes this precise.

9.2 Statement of the exclusion theorem

Theorem 9.1 (Elliptic complex exclusion **[Established]**)**.** *Let A be a commutative, associative $\mathcal{R}_\partial$-algebra with* $\operatorname{rank}_{\mathcal{R}_\partial} A = 2$*, presented as*

$$A \;\cong\; \mathcal{R}_\partial[i]/(i^2 + 1),$$

*i.e. with the elliptic quadratic relation $i^2 = -1$, working throughout over the localisation $\mathcal{R}_\partial[1/2]$ (so that 2 is invertible and the halved expressions of the argument are integral; cf. Remark **??** of §**??**). Then:*

(1) (Local integral domain at inert primes.) *If $\mathfrak{p} \subset \mathcal{R}_\partial$ is a prime of the primorial CRT decomposition with residue field $k := \mathcal{R}_\partial/\mathfrak{p}$ satisfying $-1 \notin k^{\times 2}$, then the local factor $A_\mathfrak{p} := A \otimes_{\mathcal{R}_\partial} k \cong k[i]/(i^2 + 1)$ is a field, hence an integral domain. This is a preliminary local observation; it does* not *propagate to a global integral-domain statement for A, because by Dirichlet's theorem infinitely many primes $p \equiv 1 \pmod 4$ occur in the primorial ladder, and at each such prime the local factor $\mathbb{F}_p[i]/(i^2 + 1) \cong \mathbb{F}_p \times \mathbb{F}_p$* splits *and therefore carries non-trivial local idempotents.*

(2) (No σ-equivariant swapped idempotent pair.) *In $A = \mathcal{R}_\partial[i]/(i^2 + 1)$ (over $\mathcal{R}_\partial[1/2]$), no σ-equivariant swapped idempotent pair exists. Concretely, if one attempts to build a pair $e_\pm = \frac{1}{2}(1 \pm \xi)$ from a σ_{ell}-antifixed element $\xi \in A$ (i.e. $\sigma_{\text{ell}}(\xi) = -\xi$), the elliptic involution forces $\xi \in \mathcal{R}_\partial \cdot i$, and any such $\xi = c \cdot i$ satisfies $\xi^2 = -c^2$. The orthogonality requirement $e_+ e_- = \frac{1}{4}(1 - \xi^2) = \frac{1}{4}(1 + c^2) = 0$ has no solution in $\mathcal{R}_\partial[1/2]$ because $1 + c^2 \neq 0$ for every $c \in \mathcal{R}_\partial[1/2]$ (the localised boundary ring embeds in $\mathbb{R}$ stagewise, where $1 + c^2 \geq 1 > 0$). The canonical candidate $\xi = i$ gives $e_+ e_- = \frac{1}{4}(1 - i^2) = \frac{1}{4}(1 - (-1)) = \frac{1}{2} \neq 0$. Hence no pair of non-trivial orthogonal idempotents in A is swapped by σ_{ell}.*

(3) (Incompatibility with B/C bipartition.) *Consequently, A admits no $\mathcal{R}_\partial$-linear decomposition $1 = p_B + p_C$ into orthogonal idempotents $p_B p_C = 0$, $p_B^2 = p_B$, $p_C^2 = p_C$, $p_B, p_C \notin \{0, 1\}$ that is simultaneously σ_{ell}*-equivariant *(exchanging the two idempotents). The prime-polarity bipartition of [**?**] requires such a σ-equivariant pair (axiom (C4) of Definition **??**); it cannot be realised inside A as a pair of algebraic projectors compatible with the canonical involution. Any local idempotents*

*that do exist at primes $p \equiv 1 \pmod 4$ (from part **??**) lie in CRT-local factors $A_{\mathfrak{p}}$ and are* not *σ-equivariant in the globally-required sense.*

(4) (Involution is a rotation, not a polarity swap.) *The canonical involution $\sigma_{\text{ell}} : i \mapsto -i$ on A has fixed set $A^{\sigma_{\text{ell}}} = \mathcal{R}_\partial \cdot 1$ (the scalar line only) and acts on the complementary imaginary line $\mathcal{R}_\partial \cdot i$ by negation. No nontrivial fixed subspace exists in the imaginary part, and no non-trivial idempotents are swapped by σ_{ell}. It is therefore a Galois rotation of order* 2, not *a polarity swap in the sense of Hinge 3 [?].*

(5) (Contrast with the split-complex case.) *The split-complex algebra $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2 - 1)$ is* not *an integral domain — it contains the zero divisors $(1 + j)(1 - j) = 0$ — and carries exactly two non-trivial idempotents $e_+ = \frac{1}{2}(1 + j)$, $e_- = \frac{1}{2}(1 - j)$. The canonical involution $\sigma_{\text{split}} : j \mapsto -j$ fixes $\mathcal{R}_\partial \cdot 1$ pointwise* and *swaps the idempotent pair $e_+ \leftrightarrow e_-$, realising the polarity swap of Hinge 3 on the nose.*

Consequently the τ-kernel forces $j^2 = +1$, not $i^2 = -1$: any boundary algebra compatible with the B/C bipartition and the σ-polarity involution must be split-complex. □

Part **??** is a preliminary local observation about those primes of the primorial ladder where -1 is a quadratic non-residue; it is *not* the load-bearing step, since by Dirichlet infinitely many primes $p \equiv 1 \pmod 4$ appear in the ladder, and at each such prime the local factor of A does carry non-trivial idempotents. The main structural obstruction is part **??**: even in the presence of those local idempotents, no globally σ_{ell}-equivariant swapped pair of non-trivial orthogonal idempotents can be assembled in A. Part **??** extracts the incompatibility with the B/C bipartition; part **??** is the representation-theoretic obstruction (the canonical involution is a Galois rotation, not a polarity swap); part **??** verifies that $\mathbb{D}$ passes the same test negatively, i.e. is exactly what an idempotent-carrying two-dimensional algebra with a polarity-swap involution looks like. We prove each part in turn in §**??**.

Illustrative computation: halving the unit versus halving the split-unit

The structural difference between parts **??** and **??** becomes transparent at the level of halving $(1 + \text{unit})$. In the elliptic algebra $\mathcal{R}_\partial[i]/(i^2 + 1)$,

$$\Big(\frac{1+i}{2}\Big)^2 \;=\; \frac{1+2i+i^2}{4} \;=\; \frac{2i}{4} \;=\; \frac{i}{2} \;\neq\; \frac{1+i}{2},$$

so $(1 + i)/2$ is *not* idempotent — it is a scaled rotation of the plane (a scaled $\frac{\pi}{4}$ Cayley generator). By contrast, in the split-complex algebra $\mathcal{R}_\partial[j]/(j^2 - 1)$,

$$\Big(\frac{1+j}{2}\Big)^2 \;=\; \frac{1+2j+j^2}{4} \;=\; \frac{2+2j}{4} \;=\; \frac{1+j}{2} \;=\; e_+,$$

so $(1 + j)/2$ *is* idempotent, and equals the canonical B-lobe projector. The algebraic signature $j^2 = +1$ versus $i^2 = -1$ is exactly what distinguishes *idempotent halving* from *rotation halving*: the $+1$ sign substitutes back into $1 + 2j + 1 = 2(1 + j)$, cancelling a factor of 2 and landing on the halved element; the -1 sign substitutes back into $1 + 2i - 1 = 2i$, cancelling the constant term and rotating out of the halved element. No pair of halved-unit elements in the elliptic algebra can simultaneously satisfy $x^2 = x$; the $(1 \pm \xi)/2$ template can yield an idempotent pair *only* when $\xi^2 = +1$.

9.3 Proofs of parts (1)–(5)

Proof of **??**. By the constructive CRT of Theorem **??**–Corollary **??**, every prime $\mathfrak{p}$ of $\mathcal{R}_\partial$ arising from the primorial ladder has residue field $k \cong \mathbb{Z}/p\mathbb{Z}$ for some rational prime p in the ladder. The local factor of A at $\mathfrak{p}$ is

$$A_{\mathfrak{p}} \;\cong\; k[i]/(i^2 + 1).$$

Suppose -1 is not a square in $k^\times$. Then the polynomial $X^2 + 1 \in k[X]$ has no root in k, so it is irreducible of degree 2, and $k[i]/(i^2 + 1)$ is the finite quadratic field extension $k(\sqrt{-1})$. A field has no zero divisors, hence $A_{\mathfrak{p}}$ is an integral domain.

Concretely, for $k = \mathbb{Z}/p\mathbb{Z}$ with p an odd prime, the first supplementary law of quadratic reciprocity gives $-1 \in (\mathbb{Z}/p\mathbb{Z})^{\times 2}$ iff $p \equiv 1 \pmod 4$. For every prime $p \equiv 3 \pmod 4$ in the primorial, the hypothesis holds and the local factor is $\mathbb{F}_{p^2}$, a field (cf. [?, Ch. I, §1]; [?, Ch. I, §2]). At the complementary primes $p \equiv 1 \pmod 4$, by contrast, -1 *is* a square in k, so $X^2 + 1$ splits as $(X - \lambda)(X + \lambda)$ for some $\lambda \in k$ with $\lambda^2 = -1$, and the local factor $k[i]/(i^2 + 1) \cong k \times k$ is *not* a domain; it carries the local CRT idempotents $\frac{1}{2}(1 \pm \lambda^{-1} i)$. By Dirichlet's theorem, infinitely many such primes appear in the primorial ladder, so the "every local factor is a domain" hypothesis is never met globally for A. □

Proof of **??**. We argue directly that the two-dimensional $\mathcal{R}_\partial[1/2]$-algebra $A = \mathcal{R}_\partial[i]/(i^2+1)$ carries no σ_{ell}-equivariant swapped pair of non-trivial orthogonal idempotents — which is the structural obstruction at play, irrespective of the local idempotents identified in **??** above.

Any such pair must have the form $e_\pm = \frac{1}{2}(1 \pm \xi)$ for some element $\xi \in A$ with $\xi^2 \in \mathcal{R}_\partial[1/2] \cdot 1$ and $\sigma_{\text{ell}}(\xi) = -\xi$ (so that the pair is swapped by σ_{ell}), because: writing $e_+ = a + bi$ and requiring $\sigma_{\text{ell}}(e_+) = e_-$ forces $\sigma_{\text{ell}}(e_+) = a - bi = 1 - e_+ = (1-a) - bi$, i.e. $a = 1/2$; hence $e_+ = \frac{1}{2} + bi = \frac{1}{2}(1 + 2bi)$, and $\xi = 2bi$ is then antifixed by σ_{ell}.

With $\xi = c \cdot i$ for $c := 2b \in \mathcal{R}_\partial[1/2]$, we compute

$$\xi^2 \;=\; c^2 \cdot i^2 \;=\; -c^2 \qquad (c \in \mathcal{R}_\partial[1/2]),$$

so $\xi^2 = -c^2 \in \mathcal{R}_\partial[1/2] \cdot 1$ automatically; the idempotency equation $e_+^2 = e_+$ unfolds to

$$\tfrac{1}{4}(1 + 2\xi + \xi^2) \;=\; \tfrac{1}{2}(1 + \xi),$$

equivalently $1 + \xi^2 = 2 \cdot 1$, i.e. $\xi^2 = 1$. Combined with $\xi^2 = -c^2$, this forces $-c^2 = 1$, i.e. $c^2 = -1$ in $\mathcal{R}_\partial[1/2]$. But $\mathcal{R}_\partial[1/2]$ embeds stagewise into products of residue fields $\mathbb{F}_p$ of *real* characteristic, and in the connected identity component (cf. the global-connectedness argument of Theorem **??**, Step 3, and of Corollary **??**) the scalar subring admits no solution to $c^2 = -1$ — indeed, $\mathcal{R}_\partial[1/2]$ is an inverse limit of rings $\mathbb{Z}[1/2]/M_k\mathbb{Z}[1/2]$ in which -1 is not a global square (as a compatible tower, a square root of -1 would yield a square root of -1 in $\mathbb{F}_p$ for *every* prime p in the ladder, contradicting $-1 \notin \mathbb{F}_3^{\times 2}$ at $p = 3$).

Equivalently, without passing through $\xi^2 = 1$, one may observe the orthogonality directly: with $\xi = c \cdot i$,

$$e_+ e_- \;=\; \tfrac{1}{4}(1-\xi)(1+\xi) \;=\; \tfrac{1}{4}(1 - \xi^2) \;=\; \tfrac{1}{4}(1 + c^2) \;=\; 0$$

has no solution in $\mathcal{R}_\partial[1/2]$ because $1 + c^2 \neq 0$. The canonical candidate $\xi = i$ (i.e. $c = 1$) gives the concrete obstruction displayed in the statement of part **??**:

$$e_+ e_- \;=\; \tfrac{1}{4}(1 - i^2) \;=\; \tfrac{1}{4}(1 - (-1)) \;=\; \tfrac{1}{2} \;\neq\; 0.$$

Hence no σ_{ell}-equivariant swapped idempotent pair exists in A.

Remark on the local idempotents of part **??**. At a prime $p \equiv 1 \pmod 4$, the local factor $k[i]/(i^2+1) \cong k \times k$ does have the non-trivial local idempotents $\frac{1}{2}(1 \pm \lambda^{-1} i)$ where $\lambda \in k$ is a local square root of -1. These idempotents are *not* σ_{ell}-equivariant: σ_{ell} fixes the scalars of k but negates i, so $\sigma_{\text{ell}}(\frac{1}{2}(1 + \lambda^{-1} i)) = \frac{1}{2}(1 - \lambda^{-1} i)$. However λ is only locally defined (only $p \equiv 1 \pmod 4$ admits a square root of -1, and the global lift $\lambda \in \mathcal{R}_\partial[1/2]$ does not exist by the preceding argument), so the local pair does not assemble into a globally σ_{ell}-equivariant decomposition of $1 \in A$. □

Proof of **??**. Suppose for contradiction that $1 = p_B + p_C$ in A with $p_B p_C = 0$, $p_B^2 = p_B$, $p_C^2 = p_C$, $p_B, p_C \notin \{0, 1\}$, and $\sigma_{\text{ell}}(p_B) = p_C$ (this final requirement is the polarity-swap axiom (C4) of Definition **??**: the involution that exchanges the two lobes must exchange the two corresponding idempotents). Setting $\xi := p_B - p_C$ gives an element with $\sigma_{\text{ell}}(\xi) = p_C - p_B = -\xi$, and the idempotent pair takes the form $p_\pm = \frac{1}{2}(1 \pm \xi)$, contradicting part **??**. Therefore no such σ_{ell}-equivariant decomposition exists.

The axiom (C4) cannot be dropped: the prime-polarity bipartition of [**?**] is, by construction, the fixed set of the polarity-swap involution, and the B-lobe idempotent is carried to the C-lobe idempotent by that very involution. Any attempt to realise the bipartition inside A without σ_{ell}-equivariance must therefore import the involution from *outside* A — e.g. as a coordinate choice on $\mathcal{R}_\partial \oplus \mathcal{R}_\partial \cdot i$ — which is exactly the *unearned diagonal* forbidden by the boundary-first discipline of the hinge programme [**?**, **?**]. Consequently, no $\mathcal{R}_\partial$-linear bipartition of A into algebraic projectors compatible with σ_{ell} is available, and the prime-polarity bipartition cannot be realised inside A as a pair of algebraic projectors of the required kind. □

Proof of **??**. Write an arbitrary element of A as $x = a + bi$ with $a, b \in \mathcal{R}_\partial$. The canonical involution is

$$\sigma_{\text{ell}}(a + bi) \;=\; a - bi.$$

A fixed point satisfies $a + bi = a - bi$, i.e. $2bi = 0$. In A, the element i is a unit (indeed $i \cdot (-i) = 1$), and 2 is invertible away from the ramified prime $\mathfrak{p} = (2)$; hence $b = 0$ on the unramified part, and $A^{\sigma_{\text{ell}}} = \mathcal{R}_\partial \cdot 1$. The complementary eigenspace $A^{-\sigma_{\text{ell}}} = \mathcal{R}_\partial \cdot i$ carries σ_{ell} by negation. Thus σ_{ell} is a $\mathbb{Z}/2\mathbb{Z}$-grading whose non-trivial component is the imaginary line, with no fixed geometric structure inside the non-scalar part. This is the classical Galois action of $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ on the elliptic quadratic extension; geometrically a rotation by π of the imaginary axis.

Now compare with the split-complex algebra $\mathbb{D}$. Writing $x = a + bj$ and $\sigma_{\text{split}}(a + bj) = a - bj$, the same fixed-set calculation gives $\mathbb{D}^{\sigma_{\text{split}}} = \mathcal{R}_\partial \cdot 1$; so far the two involutions look alike. But σ_{split} has an additional feature absent in the elliptic case: it swaps the non-trivial idempotents,

$$\sigma_{\text{split}}(e_+) \;=\; \sigma_{\text{split}}(\tfrac{1}{2}(1 + j)) \;=\; \tfrac{1}{2}(1 - j) \;=\; e_-, \qquad \sigma_{\text{split}}(e_-) \;=\; e_+.$$

This is the *polarity swap*: an involution that exchanges the B-lobe idempotent with the C-lobe idempotent, realising the σ-symmetry identified in Hinge 3 as the defining feature of the prime-polarity involution. The elliptic involution σ_{ell} has no such swap to perform, because there are no non-trivial idempotents to exchange (part **??**). Qualitatively:

Involution	Fixed set	Acts on non-scalar part by
$\sigma_{\text{ell}} : i \mapsto -i$	$\mathcal{R}_\partial \cdot 1$	rotation (negation of imaginary line)
$\sigma_{\text{split}} : j \mapsto -j$	$\mathcal{R}_\partial \cdot 1$	polarity swap $e_+ \leftrightarrow e_-$

The two involutions share a fixed-set description but differ in what they do to the idempotent structure; an involution is a rotation in the elliptic case and a polarity swap in the split-complex case, and these are not the same algebraic datum. □

Proof of **??**. In $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2 - 1)$, compute

$$(1 + j)(1 - j) \;=\; 1 - j^2 \;=\; 1 - 1 \;=\; 0,$$

so $\mathbb{D}$ has a zero divisor and is not an integral domain; this is the classical presentation of the hyperbolic number plane [**?**, **?**, **?**]. The idempotents $e_+ = \frac{1}{2}(1 + j)$ and $e_- = \frac{1}{2}(1 - j)$ satisfy

$$e_+^2 = e_+, \qquad e_-^2 = e_-, \qquad e_+e_- = \tfrac{1}{4}(1 + j)(1 - j) = 0, \qquad e_+ + e_- = 1,$$

exhibiting the canonical B/C decomposition $\mathbb{D} \cong \mathcal{R}_\partial \cdot e_+ \times \mathcal{R}_\partial \cdot e_- \cong \mathcal{R}_\partial \times \mathcal{R}_\partial$ required by Theorem **??**. Over any local factor $\mathcal{R}_\partial/\mathfrak{p}$ (odd residue characteristic) the reduction of $j^2 - 1$ splits as $(X - 1)(X + 1)$, so $\mathbb{D}_\mathfrak{p} \cong (\mathcal{R}_\partial/\mathfrak{p}) \times (\mathcal{R}_\partial/\mathfrak{p})$ and the idempotent structure persists stagewise; stabilisation of this structure across the primorial ladder is the content of Theorem **??**. Thus $\mathbb{D}$ realises the B/C decomposition and the polarity swap algebraically, in sharp contrast to the elliptic case covered by parts **??**–**??**. □

Corollary 9.2 (Canonical uniqueness of the τ-boundary algebra **[τ-Effective]**). *Among the three quadratic two-dimensional commutative $\mathcal{R}_\partial$-algebras (elliptic, parabolic, split-complex), only $\mathbb{D}$ admits a pair of non-trivial orthogonal idempotents realising the B/C bipartition, a polarity-swap involution fixing the scalar line, and a balanced mediator ι_τ at the junction of the two lobes. Theorems* **??** *and* **??**, *combined with Theorem* **??**, *therefore pin down $\mathbb{D}$ as the canonical host algebra for the boundary characters: no alternative is available.*

Proof. The elliptic case is ruled out by Theorem **??**. The parabolic case $A \cong \mathcal{R}_\partial[\varepsilon]/(\varepsilon^2)$ is ruled out because the only idempotents of a ring with nilpotents $\varepsilon \neq 0, \varepsilon^2 = 0$ are $\{0, 1\}$: $x^2 = x$ with $x = a + b\varepsilon$ gives $a^2 = a$ in $\mathcal{R}_\partial$ and $2ab = b$, so $b(2a - 1) = 0$ with $2a - 1$ a unit, hence $b = 0$. Only case (iii), $\mathbb{D}$, passes. □

9.4 Historical context: Minkowski, Yaglom, Sobczyk

The split-complex algebra, often written $\mathbb{R}[j]$ with $j^2 = +1$, has been rediscovered independently under several names: *hyperbolic numbers*, *perplex numbers*, *double numbers*, and *Minkowski numbers*. Yaglom's monograph [?] develops it as the natural scalar substrate of the Minkowski plane, with the light-cone coordinates playing the role of our idempotents e_+, e_-; Sobczyk's expository article [?] formulates it directly as the *hyperbolic number plane*, emphasising the idempotent decomposition as a feature (not a defect); Catoni et al. [?] carry out the full geometric and analytic theory of Minkowski spacetime in these terms. In each case the authors are explicit that split-complex algebras fail to be integral domains, and that this failure is exactly what allows them to host the two null (light-cone) directions of Minkowski space as independent algebraic factors.

The same structural dichotomy appears in the classification of real Clifford algebras $C\ell_{p,q}(\mathbb{R})$: the elliptic case $C\ell_{0,1}(\mathbb{R}) \cong \mathbb{C}$ is an integral domain with no idempotents, while the hyperbolic case $C\ell_{1,0}(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}$ (via the split-complex isomorphism $\mathbb{D} \cong \mathbb{R} \times \mathbb{R}$) decomposes into two orthogonal real factors. Our Theorem ?? is the boundary-ring analogue of this classical Clifford dichotomy: elliptic quadratic extensions are fields (domains, no idempotents); hyperbolic quadratic extensions split as products (non-domains, two idempotents). What is novel here is not the algebra itself but the *forcing*: in our setting the B/C prime bipartition is a primitive piece of data, so the non-domain side is canonically selected rather than elected.

9.5 Structural moral: idempotents as features, not defects

A recurring motif in orthodox exposition treats the zero divisors of $\mathbb{D}$ as a pathology: "$\mathbb{D}$ is a pathological ring; real division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$." This is correct in the category of division algebras, but the category is the wrong one for boundary characters. What the τ-kernel *needs* is not division but *idempotent decomposition*: an algebraic encoding of the fact that the boundary has two lobes, that the prime spectrum splits in two halves, and that characters therefore factor through a two-projector decomposition. Zero divisors and non-trivial idempotents are two sides of the same coin: $e_+e_- = 0$ says that the idempotents are disjoint, and it equally says that $\mathbb{D}$ is not a domain.

Theorem ?? should therefore be read as follows: *the algebraic property the boundary algebra is required to have (a B/C idempotent decomposition) is exactly the property the elliptic algebra fails to have, and is exactly the property the split-complex algebra has canonically.* What looks like a defect of $\mathbb{D}$ from the viewpoint of division algebras is the feature that makes it the unique host for the boundary characters. This is the structural reason behind the corrected Path-A derivation of ι_τ: the $(2/p)$ Legendre classifier (mod 8), which lives in the split-complex setting, is the τ-native invariant, whereas the Gaussian classifier $(-1/p)$ (mod 4), which lives in the elliptic Gaussian integers $\mathbb{Z}[i]$, inherits the structural incompatibility of Theorem ?? and cannot serve as the τ-native classifier.

9.6 Numerical illustration: mod-4 vs mod-8 on small primes

The structural obstruction of Theorem ?? is visible numerically in the disagreement between the elliptic Gaussian classifier $\mathrm{Gauss}(p) := (-1/p) = (-1)^{(p-1)/2}$ (which sorts primes mod 4) and the split-complex Legendre classifier $\mathrm{Leg}(p) := (2/p) = (-1)^{(p^2-1)/8}$ (which sorts primes mod 8). The two classifiers agree on some primes and disagree on others; only the second is consistent with the τ-kernel's B/C bipartition ([?]). Table ?? illustrates.

The disagreement is structural, not an accident of small primes: Gauss is periodic mod 4 and Leg is periodic mod 8, so any prime p with $p \equiv 5 \pmod 8$ (hence $p \equiv 1 \pmod 4$) has $\mathrm{Gauss}(p) = +1$ but $\mathrm{Leg}(p) = -1$, and any prime $p \equiv 7 \pmod 8$ (hence $p \equiv 3 \pmod 4$) has $\mathrm{Gauss}(p) = -1$ but $\mathrm{Leg}(p) = +1$. By Dirichlet's theorem each of these residue classes contains infinitely many primes, so the disagreement set is itself infinite, of natural density $1/2$ within the primes. This is the numerical face of Theorem ??: the elliptic Gaussian classifier is not merely mildly suboptimal but structurally the wrong invariant at an infinite family of primes, and the v3.3 $\to$ v4 correction of ι_τ (Path A, [?]) is precisely the replacement of Gauss by Leg forced by the idempotent structure of $\mathbb{D}$.

Remark 9.3 (Why this is **[Established]** scope)**.** Part ?? of Theorem ?? uses nothing beyond standard commutative algebra over a profinite ring and the first supplementary law of quadratic reciprocity; it is **[Established]** without reliance on τ-kernel machinery. Part ?? — the σ-equivariant swapped-idempotent obstruction — is likewise **[Established]**: it is the elementary linear algebra of rank-2 commutative algebras with involution, together with the global non-squareness of -1 in the profinite scalar subring $\mathcal{R}_\partial[1/2]$ (a consequence of Corollary ?? via Dirichlet's theorem). Part ?? uses the B/C bipartition as primitive input from [?], but once that bipartition is accepted as data together with its defining σ-equivariance, the no-go conclusion is again pure algebra. Parts ??–?? are direct computations inside the two quadratic algebras. The **[τ-Effective]** content of the paper

p	p mod 4	Gauss$(p) = (-1/p)$	p mod 8	Leg$(p) = (2/p)$	τ-class
3	3	-1	3	-1	C
5	1	$+1$	5	-1	C
7	3	-1	7	$+1$	B
11	3	-1	3	-1	C
13	1	$+1$	5	-1	C
17	1	$+1$	1	$+1$	B
19	3	-1	3	-1	C
23	3	-1	7	$+1$	B
29	1	$+1$	5	-1	C
31	3	-1	7	$+1$	B

Table 1. The elliptic Gaussian classifier $(-1/p)$ (mod 4) and the split-complex Legendre classifier $(2/p)$ (mod 8) disagree on primes such as $p \in \{5, 7, 13, 23, 29, 31\}$. The rightmost column records the τ-class B (resp. C) assigned by the split-complex character $\tilde{\chi}$ of [**?**]; it agrees with Leg by construction and disagrees with Gauss precisely on those primes where the mod-4 and mod-8 classes split apart.

resides in the *identification* of the B/C bipartition with the prime-polarity bipartition of the τ-kernel (Hinge 2), not in the exclusion theorem itself. This is why the theorem is classifiable as **[Established]**.

10. THE FOUR-ATOM SPECTRAL DICTIONARY

The split-complex boundary algebra $\mathbb{D}$ constructed in §**??** carries four canonical σ-equivariant idempotents. We show in this section that these four canonical idempotents are in bijection with four *channel-eigenstates* of the lemniscate boundary, that the σ-involution (bipolar swap) on $\mathbb{D}$ matches the charge-conjugation involution on those channel-eigenstates, and that the unique non-idempotent σ-fixed balanced scalar — the crossing mediator ι_τ of §**??** — plays the role of the ω-generator (Higgs sector) mediating the two polarised lobes. The result is a clean structural match between the algebraic skeleton of $\mathbb{D}$ and the 2nd-Edition generator naming of Category τ [**?**]: a four-atom dictionary with one crossing mediator, realising what Book I calls the 4+1 *sector structure* of the physics stratum.

10.1 The canonical σ-equivariant Boolean sublattice

Recall from Definition **??** and Theorem **??** that

$$\mathbb{D} \;=\; \{a + bj \mid a, b \in \mathcal{R}_\partial,\; j^2 = +1\} \;\cong\; \mathcal{R}_\partial \times \mathcal{R}_\partial$$

with canonical orthogonal idempotents

$$e_+ \;=\; \tfrac{1}{2}(1+j), \qquad e_- \;=\; \tfrac{1}{2}(1-j),$$

satisfying $e_+^2 = e_+$, $e_-^2 = e_-$, $e_+e_- = 0$, and $e_+ + e_- = 1$. We work throughout over the localisation $\mathcal{R}_\partial[1/2]$ of Remark **??**, so that the halved expressions above are integral; this convention is in force for *every* idempotent and projector statement of the present section and we drop the $[1/2]$ from the notation.

The dictionary we are about to construct matches four canonical idempotents with four channel-eigenstates. The full idempotent set Idem$(\mathbb{D})$ is, however, very much larger than four (see Remark **??** below): by the constructive CRT of Corollary **??**, $\mathcal{R}_\partial$ has profinitely many idempotents indexed by the prime ladder, and $\mathbb{D} \cong \mathcal{R}_\partial \times \mathcal{R}_\partial$ inherits at least as many. The four canonical idempotents of the dictionary are those forced by the split-complex structure itself — by the relation $j^2 = +1$ and the σ-involution that exchanges the two lobes — not by the prime-by-prime decomposition of the base ring. We now isolate them as the *σ-equivariant Boolean sublattice* $B_\sigma(\mathbb{D})$ of Idem$(\mathbb{D})$ and prove that it has exactly four elements.

Definition 10.1 (σ-equivariant Boolean sublattice)**.** *Write $\sigma : \mathbb{D} \to \mathbb{D}$ for the $\mathcal{R}_\partial$-algebra involution $\sigma(j) = -j$. Define $B_\sigma(\mathbb{D}) \subseteq$ Idem$(\mathbb{D})$ to be the* smallest Boolean sub-$*$-algebra of idempotents *closed under σ and containing the canonical* *pair* $\{e_+, e_-\}$*. Concretely,* $B_\sigma(\mathbb{D})$ *is the intersection of all Boolean sublattices* $B \subseteq \operatorname{Idem}(\mathbb{D})$ *with* $0, 1 \in B$*,* $\sigma(B) = B$*, and* $e_+, e_- \in B$*, where Boolean-lattice operations are* $e \wedge f = e \cdot f$*,* $e \vee f = e + f - ef$*,* $\neg e = 1 - e$*.*

Lemma 10.2 (Four canonical σ-equivariant idempotents **[Established]**)**.** *Within* $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2-1)$*, the* σ*-equivariant Boolean sublattice of idempotents has exactly four elements:*

$$B_\sigma(\mathbb{D}) \;=\; \{0,\; e_+,\; e_-,\; 1\}.$$

The involution σ *restricted to* $B_\sigma(\mathbb{D})$ *acts by* $\sigma(0) = 0$*,* $\sigma(e_+) = e_-$*,* $\sigma(e_-) = e_+$*,* $\sigma(1) = 1$*; the two fixed elements are the trivial idempotents and the two non-fixed elements form a single* σ*-orbit of length two. These four idempotents are the canonical* atoms *of the lemniscate bipolar decomposition: the pair* (e_+, e_-) *is the unique pair of non-trivial mutually* σ*-swapping orthogonal idempotents arising from the split-complex relation* $j^2 = +1$ *itself, as opposed to the (vastly more numerous) CRT-refined idempotents arising from the prime-by-prime decomposition of* $\mathcal{R}_\partial$*.*

Proof. We work over $\mathcal{R}_\partial[1/2]$ throughout (so that $e_+ = \frac{1}{2}(1+j)$ and $e_- = \frac{1}{2}(1-j)$ are integral). Using the basis $\{1, j\}$ of $\mathbb{D}$ over $\mathcal{R}_\partial$, write a candidate idempotent as $e = a + bj$ with $a, b \in \mathcal{R}_\partial$. The idempotency equation $e^2 = e$ reads

$$(a^2 + b^2) \;+\; 2ab\,j \;=\; a \;+\; b\,j,$$

equivalent to the pair

$$a^2 + b^2 \;=\; a, \qquad 2ab \;=\; b. \tag{1}$$

We now enumerate the elements of $B_\sigma(\mathbb{D})$ by case analysis on the σ-orbit of e.

Case 1: $\sigma(e) = e$ *(fixed elements).* Fixedness gives $a - bj = a + bj$, i.e. $2b = 0$, so $b = 0$ (as 2 is invertible). Then (**??**)$_1$ reduces to $a^2 = a$ in $\mathcal{R}_\partial$. Any element of $B_\sigma(\mathbb{D})$ is, by definition, a finite Boolean combination of $\{0, 1, e_+, e_-\}$ under $\wedge, \vee, \neg$, so its scalar-part coefficient a lies in the Boolean-algebra closure of $\{0, 1\} \subset \mathcal{R}_\partial$ — which is simply $\{0, 1\}$ itself. Equivalently: the only *globally* σ*-fixed* scalar idempotents of $\mathcal{R}_\partial$ that occur in $B_\sigma(\mathbb{D})$ are the trivial ones, because the CRT-local idempotents of $\mathcal{R}_\partial$ (which exist in $\operatorname{Idem}(\mathcal{R}_\partial)$ but not in the σ-generated closure of the canonical pair) are *not* generated by $\{e_+, e_-\}$ under Boolean operations and σ. Hence $a \in \{0, 1\}$ and $e \in \{0, 1\}$.

Case 2: $\sigma(e) = 1 - e$ *(non-fixed orbit).* Non-fixed idempotents in a Boolean sublattice occur in complementary pairs; if e is non-fixed then $\sigma(e)$ is its Boolean complement if and only if $e + \sigma(e) = 1$ (equivalently, σ permutes the two non-trivial idempotents of a pair summing to 1). Setting $\sigma(e) = 1 - e$ gives $a - bj = (1 - a) - bj$, i.e. $a = 1/2$ (here using 2 invertible in $\mathcal{R}_\partial[1/2]$). Substituting $a = 1/2$ into (**??**)$_1$:

$$\tfrac{1}{4} + b^2 \;=\; \tfrac{1}{2} \qquad \Longrightarrow \qquad b^2 \;=\; \tfrac{1}{4} \qquad \Longrightarrow \qquad b \;=\; \pm\tfrac{1}{2}$$

(the square-root determination being forced by global connectedness of $\mathcal{R}_\partial$ on its identity component, i.e. $b = 1/2$ or $b = -1/2$ consistently at every stage of the primorial ladder, as guaranteed by Theorem **??** Step 3). Equation (**??**)$_2$ is automatically satisfied ($2 \cdot \frac{1}{2} \cdot b = b$). The two solutions give exactly

$$e \;=\; \tfrac{1}{2} + \tfrac{1}{2}j \;=\; e_+, \qquad e \;=\; \tfrac{1}{2} - \tfrac{1}{2}j \;=\; e_-,$$

with $\sigma(e_+) = e_-$ and $\sigma(e_-) = e_+$ by direct computation.

Case 3: other σ*-orbits.* In a Boolean sublattice closed under σ, any element e satisfies $\sigma(e) \in B_\sigma(\mathbb{D})$, and Boolean closure requires that $e \cdot \sigma(e)$ and $e + \sigma(e) - e \cdot \sigma(e)$ also lie in $B_\sigma(\mathbb{D})$; iterating, the σ-orbit of any element is a finite Boolean-generating set. If $\sigma(e) \neq e$ and $\sigma(e) \neq 1 - e$, then $\{e, \sigma(e), 1 - e, 1 - \sigma(e)\}$ are four distinct idempotents generating a Boolean sublattice strictly larger than $\{0, 1, e_+, e_-\}$, contradicting the *minimality* requirement in Definition **??**. Hence Cases 1 and 2 exhaust $B_\sigma(\mathbb{D})$.

Combining Cases 1 and 2 gives $B_\sigma(\mathbb{D}) = \{0, e_+, e_-, 1\}$. Distinctness of the four elements is immediate from the isomorphism $\mathbb{D} \cong \mathcal{R}_\partial \times \mathcal{R}_\partial$ (the four elements correspond to the pairs $(0,0), (1,0), (0,1), (1,1)$ respectively). The canonical-atom description in the statement follows by inspection of Case 2. □

Remark 10.3 ($B_\sigma(\mathbb{D})$ versus $\mathrm{Idem}(\mathbb{D})$: the CRT distinction)**.** The full idempotent set $\mathrm{Idem}(\mathbb{D})$ is much larger than $B_\sigma(\mathbb{D})$. By the profinite Chinese Remainder decomposition of Corollary **??**, $\mathcal{R}_\partial \cong \varprojlim_k \mathbb{Z}/M_k\mathbb{Z} \cong \varprojlim_k \prod_{n\le k} \mathbb{Z}/p_n\mathbb{Z}$, so $\mathcal{R}_\partial$ already carries profinitely many idempotents indexed by subsets of the prime ladder (concretely, $\mathbb{Z}/M_k\mathbb{Z}$ has 2^k idempotents, one per subset of $\{p_1, \dots, p_k\}$; e.g. $\mathbb{Z}/6\mathbb{Z}$ has four idempotents $\{0, 1, 3, 4\}$). Since $\mathbb{D} \cong \mathcal{R}_\partial \times \mathcal{R}_\partial$, the idempotent set of $\mathbb{D}$ is at least $|\mathrm{Idem}(\mathcal{R}_\partial)|^2$, hence profinitely many.

The four atoms of $B_\sigma(\mathbb{D})$ are precisely the *structural* idempotents that are σ-equivariantly canonical — the ones forced by the $j^2 = +1$ split-complex structure itself, not by the prime-by-prime decomposition of $\mathcal{R}_\partial$. Equivalently: the CRT-derived idempotents of $\mathcal{R}_\partial$ pull back into $\mathbb{D}$ as σ-fixed elements (σ acts by $j \mapsto -j$ and fixes $\mathcal{R}_\partial \cdot 1$ pointwise), but none of them swap under σ in the lobe-exchanging sense. The Boolean sublattice $B_\sigma(\mathbb{D})$ is therefore the unique minimal carrier of the lobe-swap involution together with its two non-trivial orbits, and it is this sublattice — not the full $\mathrm{Idem}(\mathbb{D})$ — that is relevant for the τ-dictionary. The CRT-refined idempotents of $\mathbb{D}$ live "below" $B_\sigma(\mathbb{D})$ in a precise sense: each element of $B_\sigma(\mathbb{D})$ decomposes across the primorial ladder into a sum of finer local CRT-idempotents, and $B_\sigma(\mathbb{D})$ is the quotient $\mathrm{Idem}(\mathbb{D})/\!\sim$ under the equivalence "agrees on the two lobes $e_+ \cdot \mathbb{D}, e_- \cdot \mathbb{D}$ modulo CRT-local refinement."

Remark 10.4 (The four-atom pattern)**.** The cardinality $|B_\sigma(\mathbb{D})| = 4$ is not an artefact of the presentation: it is the Boolean algebra $B_2 \cong \{0, 1\}^2$ of a two-factor split, and it is the minimal nontrivial σ-equivariant spectrum compatible with the bipolar boundary structure of [**?**]. The same four-atom pattern appears in the split-complex monograph literature [**?**, **?**, **?**], where the light-cone coordinates play the role of e_+, e_- and the hyperbolic conjugation plays the role of σ; and in the generator dictionary of [**?**], which records the same observation at the level of the τ-generator naming: four canonical spectral atoms plus one completion locus (ι_τ) are forced, not chosen.

10.2 Channel-eigenstates of the lemniscate

We now name four canonical structural states of the lemniscate boundary $\mathbb{L} = S^1_B \vee S^1_C$, drawing on the paired-channel theorem and canonical prime polarisation of Hinges 2–3 [**?**, **?**]: there are exactly two primitive access channels plus the unique crossing mediator ι_τ, together with the null vacuum and the non-polarised total readout.

Definition 10.5 (Channel-eigenstates of $\mathbb{L}$)**.** *A* channel-eigenstate *is a stabilised ω-tail activity pattern on $\mathbb{L}$ of one of the four canonical types:*

(i) α-***null state:*** *no channel activity; the identically zero boundary observable.*

(ii) γ-***eigenstate:*** *B-lobe dominant; the ω-tail is eventually supported on the B-polarised germ (Hinge 2, I.T05; [**?**], canonical prime polarisation).*

(iii) η-***eigenstate:*** *C-lobe dominant; the ω-tail is eventually supported on the C-polarised germ.*

(iv) α-***total state:*** *both lobes simultaneously active with the unit-sum coupling $1 = e_+ + e_-$; the non-polarised readout associated with the α-channel (gravity, diameter refinement).*

Write the four states as $\Psi_0, \Psi_\gamma, \Psi_\eta, \Psi_\alpha$ respectively.

Remark 10.6 (Physical/structural role)**.** The γ- and η-eigenstates are the *polarised* channels (Book IV Ch. 18 [**?**]; [**?**], paired-channel theorem): they carry the electroweak B/C split and are exchanged by charge conjugation. The α-null state is the trivial vacuum (no boundary activity); the α-total state is the non-polarised, σ-fixed "both lobes together" readout, which the 2nd-Edition force mapping [**?**] assigns to the gravity (α) sector because it couples to the full diameter refinement without privileging either lobe. The ω-generator is *not* among the four states of Definition **??**: it is the crossing mediator (§**??** below), not an eigenstate.

10.3 The dictionary theorem

Recall that the bipolar swap involution $\sigma : \mathbb{D} \to \mathbb{D}$ is the $\mathcal{R}_\partial$-algebra involution $\sigma(j) = -j$, induced by the lobe-swap on $\mathbb{L}$ [**?**]. Equivalently,

$$\sigma(e_+) \;=\; e_-, \qquad \sigma(e_-) \;=\; e_+, \qquad \sigma(0) \;=\; 0, \qquad \sigma(1) \;=\; 1.$$

Hence $\{0, 1\}$ is the σ-fixed part of $B_\sigma(\mathbb{D})$, and $\{e_+, e_-\}$ is a single σ-orbit of length two. (The full idempotent set $\mathrm{Idem}(\mathbb{D})$ has a much richer σ-orbit structure induced by the CRT decomposition of Corollary **??**; but only the canonical Boolean sublattice $B_\sigma(\mathbb{D})$ enters the dictionary. See Remark **??**.)

Theorem 10.7 (Four-atom spectral dictionary **[τ-Effective]**). *There is a canonical bijection*

$$\Phi_{\rm dict} \;:\; B_\sigma(\mathbb{D}) \;\xrightarrow{\sim}\; \{\Psi_0,\, \Psi_\gamma,\, \Psi_\eta,\, \Psi_\alpha\}$$

*from the canonical σ-equivariant Boolean sublattice $B_\sigma(\mathbb{D}) = \{0, e_+, e_-, 1\}$ of Lemma **??** to the four channel-eigenstates of Definition **??**, given by*

Idempotent	*Channel-eigenstate*	*Structural role*
0	Ψ_0	*α-null (no channel activity)*
e_+	Ψ_γ	*γ-eigenstate (B-lobe dominant; EM sector)*
e_-	Ψ_η	*η-eigenstate (C-lobe dominant; strong sector)*
1	Ψ_α	*α-total (both lobes simultaneous; gravity sector)*

The map $\Phi_{\rm dict}$ is determined uniquely by three matching conditions: (i) matching of idempotent structure *(zero / single-lobe projection / full unit), (ii) σ-equivariance (matching of the bipolar-swap orbit structure), and (iii) compatibility with the Book II Ch. 47 idempotent decomposition of the calibrated boundary character ring [**?**, II.D59].*

Proof. We prove existence of $\Phi_{\rm dict}$ by verifying the three matching conditions, then uniqueness by showing any map satisfying them agrees with $\Phi_{\rm dict}$.

(i) Idempotent-structure match. The channel-eigenstates of Definition **??** form a natural lattice under "channel activity":

- Ψ_0 has *no* active channel (bottom element);
- Ψ_γ activates exactly the B-lobe (single-lobe atom);
- Ψ_η activates exactly the C-lobe (single-lobe atom);
- Ψ_α activates *both* lobes simultaneously (top element).

On the algebra side, $B_\sigma(\mathbb{D}) = \{0, e_+, e_-, 1\}$ carries the canonical Boolean-lattice structure under $e \wedge f = e \cdot f$, $e \vee f = e + f - ef$, $\neg e = 1 - e$; equivalently, the product order $x \le y \iff xy = x$ gives

$$0 \;<\; e_+,\, e_- \;<\; 1,$$

with $e_+e_- = 0$, $e_+ + e_- = 1$ (Boolean complements of each other). The assignment $\Phi_{\rm dict}(0) = \Psi_0$, $\Phi_{\rm dict}(e_+) = \Psi_\gamma$, $\Phi_{\rm dict}(e_-) = \Psi_\eta$, $\Phi_{\rm dict}(1) = \Psi_\alpha$ is the unique order-isomorphism of Boolean lattices of cardinality 4 that respects single-lobe atomicity (the single-lobe idempotents e_+, e_- map to the single-lobe eigenstates Ψ_γ, Ψ_η).

(ii) σ-equivariance match. The bipolar-swap involution acts on the four canonical idempotents as

$$\sigma:\; 0 \leftrightarrow 0,\;\; 1 \leftrightarrow 1,\;\; e_+ \leftrightarrow e_-$$

(cf. Lemma **??**). On the channel-eigenstate side, the charge-conjugation involution C (which exchanges the B- and C-lobes) acts by

$$C:\; \Psi_0 \leftrightarrow \Psi_0,\;\; \Psi_\alpha \leftrightarrow \Psi_\alpha,\;\; \Psi_\gamma \leftrightarrow \Psi_\eta.$$

By [**?**], the σ-involution on $\mathbb{D}$ is the lobe-swap involution on $\mathbb{L}$ transported through the idempotent projectors. Hence $\Phi_{\rm dict}$ commutes with the involutions:

$$\Phi_{\rm dict} \circ \sigma|_{B_\sigma(\mathbb{D})} \;=\; C \circ \Phi_{\rm dict}.$$

Equivalently, the σ-fixed idempotents $\{0, 1\}$ correspond to the C-fixed (*non-polarised*) eigenstates $\{\Psi_0, \Psi_\alpha\}$, while the σ-swapped pair $\{e_+, e_-\}$ corresponds to the C-swapped (*polarised*) pair $\{\Psi_\gamma, \Psi_\eta\}$. This uniquely determines the assignment on the non-fixed orbit once condition (i) has fixed which single-lobe idempotent carries the B-lobe (namely e_+, by Theorem **??**).

(iii) Book II Ch. 47 compatibility. The calibrated boundary character ring $H_\tau^{\rm cal}$ of [**?**, II.D59] admits the canonical decomposition

$$H_\tau^{\rm cal} \;=\; e_+ \cdot H_\tau^{\rm cal} \;\oplus\; e_- \cdot H_\tau^{\rm cal} \;\cong\; A_\tau^{(B)} \times A_\tau^{(C)},$$

with $A_\tau^{(B)} := e_+ \cdot H_\tau^{\text{cal}}$ and $A_\tau^{(C)} := e_- \cdot H_\tau^{\text{cal}}$. The B-channel character ring $A_\tau^{(B)}$ is exactly the algebra of γ-eigenstate observables (the EM sector of Book IV), and $A_\tau^{(C)}$ is the algebra of η-eigenstate observables (the strong sector). Hence $\Phi_{\text{dict}}(e_+) = \Psi_\gamma$ is forced by the identification $A_\tau^{(B)} \leftrightarrow \gamma$, and dually for e_-. The values of Φ_{dict} on 0 and 1 are then fixed as the zero and unit of the product decomposition.

Uniqueness. Any bijection $\Phi' : B_\sigma(\mathbb{D}) \to \{\Psi_0, \Psi_\gamma, \Psi_\eta, \Psi_\alpha\}$ satisfying (i)–(iii) must: preserve bottom and top ($0 \mapsto \Psi_0$, $1 \mapsto \Psi_\alpha$) by (i); preserve the σ-orbit decomposition by (ii), so $\{e_+, e_-\}$ maps bijectively to $\{\Psi_\gamma, \Psi_\eta\}$; and agree with the B/C-channel assignment of (iii), so $e_+ \mapsto \Psi_\gamma$. Hence $\Phi' = \Phi_{\text{dict}}$. □

Corollary 10.8 (σ-fixed vs. σ-swapped idempotents **[τ-Effective]**)**.** *The partition of $B_\sigma(\mathbb{D})$ by the σ-action matches the physical partition of channel-eigenstates into non-polarised and polarised sectors:*

$$B_\sigma(\mathbb{D})^{\sigma=+} \;=\; \{0,1\} \;\overset{\Phi_{\text{dict}}}{\longleftrightarrow}\; \{\Psi_0, \Psi_\alpha\} \;=\; \textit{non-polarised sector (gravity / vacuum)},$$

$$B_\sigma(\mathbb{D})^{\sigma=\text{swap}} \;=\; \{e_+, e_-\} \;\overset{\Phi_{\text{dict}}}{\longleftrightarrow}\; \{\Psi_\gamma, \Psi_\eta\} \;=\; \textit{polarised sector (EM / strong)}.$$

10.4 The ω-crossing mediator

The four channel-eigenstates of Definition **??** exhaust the canonical σ-equivariant sublattice $B_\sigma(\mathbb{D})$ of $\text{Idem}(\mathbb{D})$. The ω-generator of Category τ does not lie in $B_\sigma(\mathbb{D})$; instead it plays the role of the unique *crossing mediator* between the polarised pair (e_+, e_-).

Theorem 10.9 (ω as crossing mediator **[τ-Effective]**)**.** *The unique non-idempotent, σ-fixed, tail-stabilised element $\iota_\tau \in \mathbb{D}$ of Theorem **??** is the scalar readout of the ω-generator. Explicitly:*

- *(a) $\iota_\tau \notin B_\sigma(\mathbb{D})$ (since $\iota_\tau \approx 0.3413\ldots \notin \{0,1\}$ and ι_τ is σ-fixed, so by Lemma **??** Case 1 an idempotent σ-fixed element of $B_\sigma(\mathbb{D})$ must be 0 or 1; equivalently $\iota_\tau^2 \neq \iota_\tau$ in $\mathcal{R}_\partial$);*
- *(b) $\sigma(\iota_\tau) = \iota_\tau$ (balance across the two lobes, Definition **??**(b));*
- *(c) $e_+ \cdot \iota_\tau \neq 0$ and $e_- \cdot \iota_\tau \neq 0$ (non-degeneracy, Definition **??**(a));*
- *(d) ι_τ is the unique such element in the unit ball of $\mathbb{D}$.*

Equivalently, ι_τ is the canonical mediator *linking the polarised atoms e_+ and e_-: it is the unique balanced scalar in $\mathbb{D}$ that is not a projection onto either single lobe.*

Proof. Claim (a) holds because $\iota_\tau^2 = \iota_\tau$ together with $\iota_\tau \in \mathcal{R}_\partial \subset \mathbb{D}$ (the σ-fixed locus is $\mathcal{R}_\partial \cdot 1$) would place ι_τ among the σ-fixed elements of $B_\sigma(\mathbb{D})$ — that is, in $\{0,1\}$ by Lemma **??** Case 1 — contradicting $\iota_\tau \approx 0.341304 \notin \{0,1\}$. Claim (b) is Theorem **??**, which establishes ι_τ as σ-fixed by construction (the crossing germ is the single point of the wedge $\mathbb{L} = S^1 \vee S^1$ fixed by the lobe swap). Claim (c) is Definition **??**(a). Claim (d) is the uniqueness half of Theorem **??**: any two distinct balanced non-degenerate elements would define two distinct junctions of $\mathbb{L}$, but $\mathbb{L} = S^1 \vee S^1$ has a unique junction point.

The identification with the ω-generator (Higgs sector) follows from the 2nd-Edition force mapping [**?**]: the locked assignment $\omega \leftrightarrow$ Higgs is the correspondence between the structural role of "unique crossing mediator that couples to both polarised lobes simultaneously without being either" and the physical role of the Higgs channel. This matches the paired-channel theorem of [**?**]: there are exactly two primitive access channels plus the unique crossing mediator ι_τ. □

Remark 10.10 (Why ω is not a fifth canonical idempotent)**.** One might ask whether ω should be an additional canonical idempotent of $\mathbb{D}$. It cannot: by Lemma **??**, the σ-equivariant Boolean sublattice $B_\sigma(\mathbb{D})$ has exactly four elements, and by Theorem **??**(a), ι_τ is not among them. The ω-generator is structurally different from α, γ, η: the latter three are *projections* onto boundary sectors (they act by killing or preserving lobe activity), while ω is a *scalar mediator* that balances between sectors. In the language of Boolean algebras of idempotents vs. their underlying ring: $B_\sigma(\mathbb{D})$ is a two-element Boolean square, while ι_τ lives in the continuous (scalar) fibre above the σ-fixed point of that square. (The larger idempotent set $\text{Idem}(\mathbb{D})$ — which as noted in Remark **??** is profinitely infinite — likewise does not contain ι_τ, because $\iota_\tau^2 \neq \iota_\tau$ in $\mathcal{R}_\partial$ by Definition **??**; the point here is not that ι_τ fails to be *any* idempotent but that it fails to be one of the four *canonical* ones.)

10.5 Force-mapping tabulation and the $4+1$ sector structure

Combining Theorem **??** with the locked 2nd-Edition force mapping [**?**] yields the explicit $\mathbb{D}$-to-physics dictionary:

Idempotent	**Generator**	**Channel-eigenstate**	**Physical sector**
0	—	Ψ_0	vacuum / no channel
e_+	γ	Ψ_γ	electromagnetic (EM)
e_-	η	Ψ_η	strong
1	α	Ψ_α	gravitational
Outside $B_\sigma(\mathbb{D})$, in the σ-fixed scalar fibre:			
ι_τ	ω	— (mediator)	Higgs
Outside $\mathbb{D}$, at the base-refinement level:			
—	π	— (base refinement)	weak

Remark 10.11 (The $4+1$ asymmetry)**.** Category τ has *five* generators, but the canonical sublattice $B_\sigma(\mathbb{D})$ has only *four* idempotents. The apparent asymmetry is the signature of the $4+1$ sector structure of the Panta Rhei physics stratum:

- The four canonical idempotents $\{0, e_+, e_-, 1\}$ correspond to the four boundary-projection generators $\{\text{vacuum}, \gamma, \eta, \alpha\}$ — projections onto lobe activity.
- The crossing mediator ι_τ corresponds to ω (Higgs) — a scalar in $\mathbb{D}$ that is not idempotent.
- The fifth generator π (weak) is *not* a boundary projection and does not live in $\mathbb{D}$: it is a *base refinement* generator, acting on the base of the fibration $\tau^3 = \tau^1 \times_f T^2$ rather than on the lemniscate boundary [**?**]. It therefore has no counterpart among the canonical idempotents of the boundary algebra.

This is the structural origin of the electroweak asymmetry observed in Book IV: EM (γ) lives on the boundary as the e_+-projection, while weak (π) lives at the base-refinement level and enters the boundary algebra only through the mixing adapter (Book IV paired-channel theorem; [**?**]).

10.6 Corollary: sector decomposition of boundary observables

The dictionary delivers an immediate bridge to Book IV physics applications: every boundary observable $f \in \mathbb{D}$ decomposes canonically into its γ- and η-sector parts, and the non-polarised (gravity) component is recovered as the sum.

Corollary 10.12 (Sector decomposition **[τ-Effective]**)**.** *Every $f \in \mathbb{D}$ admits the canonical split-idempotent decomposition*

$$f \;=\; f_+e_+ + f_-e_- \qquad (f_\pm \in \mathcal{R}_\partial),$$

with projections

$$f_+ \;=\; e_+ \cdot f \in A_\tau^{(B)}, \qquad f_- \;=\; e_- \cdot f \in A_\tau^{(C)}.$$

Moreover:

(i) the γ-sector readout of f is f_+, the EM-channel component;
(ii) the η-sector readout of f is f_-, the strong-channel component;
(iii) the α-total readout (gravity / diameter) is $f_+ + f_-$, the σ-invariant part, given by $\frac{1}{2}(f + \sigma(f))$;
*(iv) the ω-readout (Higgs-scalar coupling) is given by the calibration operator $\mathrm{Cal}_\tau(f) = \iota_\tau \cdot (f_+ + f_-)$, the ι_τ-weighted gravity component (Hinge 3 ι_τ-calibration, [**?**]).*

Proof. The split decomposition $f = f_+e_+ + f_-e_-$ is Theorem **??**. Clauses (i)–(ii) are Theorem **??**, transported to observables via the characterisation $A_\tau^{(B)} = e_+ \cdot H_\tau^{\mathrm{cal}}$ and $A_\tau^{(C)} = e_- \cdot H_\tau^{\mathrm{cal}}$ ([**?**, II.D59]). Clause (iii) follows from $\sigma(e_+) = e_-$ and a direct computation:

$$\tfrac{1}{2}(f + \sigma(f)) \;=\; \tfrac{1}{2}\big((f_+e_+ + f_-e_-) + (f_+e_- + f_-e_+)\big) \;=\; \tfrac{f_++f_-}{2}(e_+ + e_-) \;=\; \tfrac{f_++f_-}{2}.$$

Clause (iv) is the calibration definition of Book II Ch. 68 (ι_τ-calibration): Cal_τ sends the σ-invariant part to its ι_τ-rescaled image, the canonical Higgs-mediated readout. ☐

Remark 10.13 (Bridge to physics applications). Corollary **??** is the algebraic skeleton used in Book IV to reconstruct sector-level fixed-point couplings: the Weinberg angle, the fine-structure constant, and the strong coupling are all computed as ι_τ-calibrated readouts of (f_+, f_-) projections of specific boundary characters (Book IV Ch. 18–22 electroweak and strong couplings [?]; Lean-certified at `TauLib.BookIV.Electroweak.WeinbergNLO`). The dictionary theorem of this section is what makes those readouts *canonical*: the B/C polarisation is not a coordinate choice but the unique σ-equivariant decomposition forced by Lemma **??** together with the Book II Ch. 47 character splitting.

Remark 10.14 (Registry identifiers). The four-atom dictionary is registered as: $\Phi_{\text{dict}} : B_\sigma(\mathbb{D}) \to \{\Psi_0, \Psi_\gamma, \Psi_\eta, \Psi_\alpha\}$: III.T87 (**[$\tau$-Effective]**); σ-equivariance of Φ_{dict}: III.T87b (**[τ-Effective]**); the canonical σ-equivariant Boolean sublattice $B_\sigma(\mathbb{D}) = \{0, e_+, e_-, 1\}$ (Lemma **??**): III.T87a (**[Established]**); $\iota_\tau \leftrightarrow \omega$ as crossing mediator: III.T87c (**[τ-Effective]**), reusing III.T86 from §**??**; sector decomposition corollary: III.T87d (**[τ-Effective]**). Formalised in `TauLib.BookIII.FourAtomDictionary`.

11. DOWNSTREAM: HOW ALL FOUR HINGES LIVE IN $\mathbb{D}$

11.1 The unifying role of $\mathbb{D}$

The preceding sections established that the split-complex boundary algebra $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2 - 1)$ is the canonical τ-admissible scalar extension of the countable profinite boundary ring $\mathcal{R}_\partial$ (Theorems **??** and **??**), and that its four idempotent atoms $\{0, e_+, e_-, 1\}$ are in canonical bijection with the τ-generators via the dictionary of §**??**. The goal of this section is to exhibit $\mathbb{D}$ as the *common algebraic home* of the four hinges [?, ?, ?] of the Panta Rhei bundle: Hyperfactorization (Hinge 1), Prime Polarity (Hinge 2), the Master Constant ι_τ (Hinge 3), and the present Boundary Algebra (Hinge 4).

Each of the three prior hinges was written as a self-contained structural theorem over $\mathbb{N}$: Hinge 1 decomposes integers into ABCD tuples, Hinge 2 classifies primes via the Legendre symbol, Hinge 3 identifies the unique σ-fixed crossing-germ scalar $\iota_\tau = 2/(\pi + e)$ on the lemniscate $\mathbb{L}$. Each object lives *prima facie* in a different ambient category. Our claim, formalised in Theorem **??** below, is that when all three are read through the boundary lens of $\mathbb{D}$, they coalesce into three faces of a single algebraic structure: $\mathbb{D}$-valued coordinate functions, $\mathbb{D}$-valued completely-additive characters, and a distinguished σ-fixed $\mathbb{D}$-scalar. Hinge 4 is, in this sense, the *structural* hinge of the bundle: the other three hinges are its projections onto the three natural $\mathbb{D}$-sector types.

Theorem 11.1 (Hinge Integration in $\mathbb{D}$ **[τ-Effective]**). *The split-complex boundary algebra $\mathbb{D}$ hosts all central objects of the Panta Rhei hinge bundle:*

1. (Hinge 1.) *The ABCD coordinate functions* $\text{coord}_A, \text{coord}_B, \text{coord}_C, \text{coord}_D : \mathbb{N}_{\geq 2} \to \mathbb{N}_{\geq 1}$ *of [?] extend canonically to* $\mathbb{D}$*-valued functions* $\widehat{\text{coord}}_* : \mathcal{R}_\partial \to \mathbb{D}$ *by idempotent-componentwise action.*
2. (Hinge 2.) *The prime polarity character* $\chi : \mathbb{P} \to \{+1, -1, 0\}$ *of [?] extends to a completely-additive* $\mathbb{D}$*-valued monoid homomorphism* $\tilde{\chi} : (\mathbb{N}, \times) \to (\mathbb{D}, +)$, *via the Hinge-3 idempotent lift of [?].*
3. (Hinge 3.) *The master constant* $\iota_\tau \in \mathbb{D}$ *is the unique* σ*-fixed non-idempotent scalar balancing* e_+ *and* e_- *in the unit ball of* $\mathbb{D}$ *— equivalently, the* ω*-atom of the four-atom dictionary of §??.*
4. (Consistency.) *The three lifts are mutually consistent: evaluating the Hinge-1 coordinates on the Hinge-2-classified primes along the primorial ladder recovers the Hinge-3 master constant in the refinement limit.*

The proof occupies §§**??**–**??**. Sections **??** and **??** discuss consequences and forward links.

11.2 Lifting Hinge 1: ABCD coordinates in $\mathbb{D}$

Recall from the Hyperfactorization paper [?] that every integer $X \geq 2$ admits a unique ABCD decomposition $X = (A \uparrow\uparrow C)^B \cdot D$, yielding the injective coordinate chart

$$\Phi_{\text{ch}} : \mathbb{N}_{\geq 2} \longrightarrow \mathbb{P} \times \mathbb{N}^3_{\geq 1}, \qquad X \longmapsto (\text{coord}_A(X), \text{coord}_B(X), \text{coord}_C(X), \text{coord}_D(X)).$$

We now lift Φ_{ch} to a $\mathbb{D}$-valued chart on the boundary ring.

Because $\mathcal{R}_\partial$ carries the canonical idempotent splitting $\mathcal{R}_\partial \cong \mathcal{R}_\partial \cdot e_+ \oplus \mathcal{R}_\partial \cdot e_-$ induced by the embedding $\mathcal{R}_\partial \hookrightarrow \mathbb{D}$ (Theorem **??**), every element $x \in \mathcal{R}_\partial$ (or more generally $x \in \mathbb{D}$) admits a unique expansion $x = x_+ e_+ + x_- e_-$ with $x_\pm \in \mathcal{R}_\partial$. Stagewise, at each primorial depth k, the components $x_\pm$ are represented by integers modulo M_k, and for sufficiently large k

the primorial reduction identifies $x_\pm$ with well-defined integer approximants in $\mathbb{N}_{\geq 1}$ (the finite-approximation witness of Lemma **??**).

Definition 11.2 (Idempotent-componentwise coordinate lift). *For $* \in \{A, B, C, D\}$, define the $\mathbb{D}$-valued coordinate*

$$\widehat{\mathrm{coord}}_* \;:\; \mathcal{R}_\partial \longrightarrow \mathbb{D}, \qquad \widehat{\mathrm{coord}}_*(x_+e_+ + x_-e_-) \;:=\; \mathrm{coord}_*(x_+)\,e_+ \;+\; \mathrm{coord}_*(x_-)\,e_-,$$

where coord_* *is applied stagewise to the primorial approximants of $x_+, x_- \in \mathcal{R}_\partial$, with the stabilised ω-tail taken as the limiting value. On ramified stages (where $x_\pm < 2$) set* coord_* *to the unit-glue convention* $\mathrm{coord}_A(1) := 0$, $\mathrm{coord}_B(1) := 0$, $\mathrm{coord}_C(1) := 0$, $\mathrm{coord}_D(1) := 1$.

Lemma 11.3 (Well-definedness of $\widehat{\mathrm{coord}}_*$). *The maps* $\widehat{\mathrm{coord}}_*$ *of Definition* **??** *are well-defined functions $\mathcal{R}_\partial \to \mathbb{D}$, compatible with the primorial refinement tower.*

Lean-grade sketch. By Theorem **??**, the decomposition $x = x_+e_+ + x_-e_-$ is unique modulo finite-witness equality. By the Hyperfactorization theorem [**?**], each coord_* is a primitive-recursive function $\mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 0}$, hence stagewise definable on the truncations $\mathcal{R}_\partial / M_k \mathcal{R}_\partial \cong \mathbb{Z}/M_k\mathbb{Z}$. Compatibility across the ladder follows because coord_* commutes with the canonical inclusion, and if $x_\pm$ stabilises in the ω-tail, so does $\mathrm{coord}_*(x_\pm)$, by the finite-witness character of the ABCD greedy peel. The unit-glue convention on $x_\pm = 1$ encodes the mediator germ of Definition **??**. □

Proposition 11.4 (Idempotent decomposition of the ABCD chart). *The lifted chart*

$$\widehat{\Phi}_{\mathrm{ch}} \;:\; \mathcal{R}_\partial \longrightarrow \mathbb{D}^4, \qquad x \longmapsto \big(\widehat{\mathrm{coord}}_A(x), \widehat{\mathrm{coord}}_B(x), \widehat{\mathrm{coord}}_C(x), \widehat{\mathrm{coord}}_D(x)\big),$$

restricts to the orthodox chart Φ_{ch} on the natural embedding $\mathbb{N}_{\geq 2} \hookrightarrow \mathcal{R}_\partial$ via the diagonal $n \mapsto n \cdot e_+ + n \cdot e_-$, and is componentwise injective on each idempotent sector. In particular, the Hinge-1 uniqueness statement (every $X \in \mathbb{N}_{\geq 2}$ has a unique ABCD tuple) lifts to: every $x \in \mathcal{R}_\partial$ admits a unique $\mathbb{D}$-valued ABCD decomposition $\widehat{\Phi}_{\mathrm{ch}}(x)$, with sector components determined independently by x_+ and x_-.

Lean-grade sketch. Restriction to the diagonal is immediate from the conventions: on $n \cdot 1 = n \cdot e_+ + n \cdot e_-$, $\widehat{\mathrm{coord}}_*(n \cdot 1) = \mathrm{coord}_*(n)\,e_+ + \mathrm{coord}_*(n)\,e_- = \mathrm{coord}_*(n) \cdot 1$. Componentwise injectivity: since e_+, e_- are orthogonal and Φ_{ch} is injective on $\mathbb{N}_{\geq 2}$ by the ABCD-injectivity corollary of [**?**], the sector maps $x_\pm \mapsto \Phi_{\mathrm{ch}}(x_\pm)$ are injective on each $\mathcal{R}_\partial$-sector. Uniqueness of the $\mathbb{D}$-valued tuple is inherited sectorwise. □

Remark 11.5 (Sector-orthogonality of the tower structure). Proposition **??** says the two boundary channels $e_+\mathcal{R}_\partial$ and $e_-\mathcal{R}_\partial$ carry independent ABCD towers. This is the *algebraic* face of the no-mixing principle of §**??**: mixing two sectors would require the tower $T(A, B, C) = (A \uparrow\uparrow C)^B$ to act across e_+/e_-, which is forbidden by $e_+e_- = 0$. Tower structure is therefore a kernel-level sector invariant, not a global property of $\mathcal{R}_\partial$.

11.3 Lifting Hinge 2: the prime polarity character in $\mathbb{D}$

Recall from the Prime Polarity paper [**?**] that the rational primes split canonically as $\mathbb{P} = \mathbb{P}_B \sqcup \mathbb{P}_C \sqcup \{2\}$ via the Legendre symbol, with $\mathbb{P}_B = \{p \equiv \pm 1 \pmod 8\}$ and $\mathbb{P}_C = \{p \equiv \pm 3 \pmod 8\}$. The prime polarity character $\chi : (\mathbb{N}, \times) \to (\mathbb{Z}, +)$ of [**?**] is completely additive with $\chi(p) = +1$ for $p \in \mathbb{P}_B$, $\chi(p) = -1$ for $p \in \mathbb{P}_C$, $\chi(2) = 0$. We now show that χ lifts canonically to a $\mathbb{D}$-valued character, and that this lift is precisely the split-complex idempotent character $\tilde{\chi}$ of [**?**].

Definition 11.6 (Split-complex polarity lift). *For $n \in \mathbb{N}$, let*

$$\nu_B(n) \;:=\; \#\{\textit{prime factors of } n \textit{ in } \mathbb{P}_B, \textit{ with multiplicity}\},$$

and let $\nu_C(n)$ denote the analogue for $\mathbb{P}_C$. Define the split-complex polarity lift *by*

$$\tilde{\chi} \;:\; (\mathbb{N}, \times) \longrightarrow (\mathbb{D}, +), \qquad \tilde{\chi}(n) \;:=\; \nu_B(n)\,e_+ \;+\; \nu_C(n)\,e_-. \tag{2}$$

Proposition 11.7 (Hinge-2 character in $\mathbb{D}$ **[τ-Effective]**). *The map $\tilde{\chi}$ of Definition **??** is a completely additive monoid homomorphism $(\mathbb{N}, \times) \to (\mathbb{D}, +)$, satisfying:*

(i) $\tilde{\chi}(mn) = \tilde{\chi}(m) + \tilde{\chi}(n)$ *and* $\tilde{\chi}(1) = 0$*;*

(ii) *on primes,* $\tilde{\chi}(p) = e_+$ *for* $p \in \mathbb{P}_B$*,* $\tilde{\chi}(p) = e_-$ *for* $p \in \mathbb{P}_C$*,* $\tilde{\chi}(2) = 0$*;*

(iii) *the signed-difference trace recovers Hinge 2's integer character:*

$$\mathrm{Tr}_-(\tilde{\chi}(n)) \ := \ \nu_B(n) - \nu_C(n) \ = \ \chi(n);$$

(iv) *the additive trace* $\mathrm{Tr}_+(\tilde{\chi}(n)) := \nu_B(n) + \nu_C(n)$ *is the non-ramified big-omega count* $\Omega^\star(n)$*, equal to the full big-omega of* n *minus the* 2*-adic valuation.*

Proof. Complete additivity of ν_B and ν_C follows from p-adic valuation, the prime-level identities follow from Definition **??**, and the trace identities are linear. The target algebra $\mathbb{D}$ is the earned $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2-1)$ of Hinge 4, not merely the symbolic $\mathbb{Z}[j]/(j^2-1)$: the factorisation $\tilde{\chi} : (\mathbb{N}, \times) \to (\mathcal{R}_\partial \cdot e_+ + \mathcal{R}_\partial \cdot e_-, +) \subset (\mathbb{D}, +)$ lands in the boundary ring sectors directly, by the primorial stabilisation of the valuation counts. □

Corollary 11.8 (The Hinge-2 partition as $\mathbb{D}$-atomic support). *Under $\tilde{\chi}$, the three Hinge-2 prime classes are in bijection with three of the four atoms of $\mathbb{D}$:*

$$\mathbb{P}_B \ \longleftrightarrow \ e_+, \qquad \mathbb{P}_C \ \longleftrightarrow \ e_-, \qquad \{2\} \ \longleftrightarrow \ 0.$$

The fourth atom $1 = e_+ + e_-$ is the neutral unit image $\tilde{\chi}(1) = 0$ in the target's additive *monoid (see Remark **??** below for the additive-vs-multiplicative distinction).*

Remark 11.9 (Three-versus-four and the completion locus). Corollary **??** realises three of the four atoms $\{0, e_+, e_-, 1\}$ as images of prime-polarity classes. The fourth atom 1 is the unit, arising as the mediator $1 = e_+ + e_-$ rather than as a polarity class; it corresponds to the *completion locus* of the dictionary of §**??**, identified there with the τ-generator α (gravity, diagonal radial closure). The Hinge-2 character only exhausts the three non-unit atoms; the fourth atom is witnessed by the σ-fixed mediator ι_τ (Hinge 3, treated next), which is the non-idempotent scalar refinement of 1 that remains balanced under $e_+ \leftrightarrow e_-$ swap.

Remark 11.10 (Three faces of the prime polarity character). Three distinct but equivalent presentations of the Hinge-2 character appear in the present programme. We name them to avoid notational confusion:

- The *pointwise polarity* $\chi_{\mathrm{pt}} : \mathbb{P} \to \{+1, -1, 0\}$ defined at each prime by $\chi_{\mathrm{pt}}(p) = +1$ if $p \in \mathbb{P}_B$, -1 if $p \in \mathbb{P}_C$, 0 at $p = 2$. This is the Legendre symbol $(2/p)$ as *set-theoretic function* on $\mathbb{P}$.
- The *signed-count character* $\chi_\Omega : (\mathbb{N}, \times) \to (\mathbb{Z}, +)$, completely additive with $\chi_\Omega(p) = \chi_{\mathrm{pt}}(p)$, so $\chi_\Omega(n) = \nu_B(n) - \nu_C(n)$. This is the Ω*-valued character* of Proposition **??**(iii), analogous to big-Ω $\Omega(n) = \sum v_p(n)$ in classical arithmetic.
- The *completely-multiplicative sign character* $\chi_{\mathrm{sgn}} : (\mathbb{N}, \times) \to (\{\pm 1\}, \times)$ recovered from χ_Ω by applying the sign map: $\chi_{\mathrm{sgn}}(n) = \mathrm{sgn}(\chi_\Omega(n))$ for n with an odd total non-ramified count, extended to $(\mathbb{N}, \times)$ by multiplicativity.

The split-complex lift $\tilde{\chi}$ of Definition **??** refines all three: it is additive in its target $(\mathbb{D}, +)$, like χ_Ω, and it retains the sector separation $(\nu_B(n)e_+ + \nu_C(n)e_-)$ that χ_Ω erases under the difference-trace. All Hinge-2 statements reduce to the corresponding statement about whichever of $\chi_{\mathrm{pt}}, \chi_\Omega, \chi_{\mathrm{sgn}}$ is convenient; in this paper we write χ for the pointwise polarity χ_{pt} and $\tilde{\chi}$ for the split-complex lift, using the other names only when disambiguation is needed.

11.4 Lifting Hinge 3: ι_τ as the ω-atom

Hinge 3 [?] identifies the master constant

$$\iota_\tau \ = \ \frac{2}{\pi + e} \ \approx \ 0.341304238875\ldots$$

as the canonical scalar readout of the unique σ-fixed crossing-germ $G_\times[\omega]$ on the lemniscate $\mathbb{L} = S^1 \vee S^1$. From the Hinge-4 vantage, ι_τ is visible in $\mathbb{D}$ directly, as the unique σ-fixed *non-idempotent* scalar balancing e_+ and e_- in the unit ball.

Definition 11.11 (ω-atom condition)**.** *An element $\zeta \in \mathbb{D}$ is an* ω-atom *(of the four-atom dictionary, §??) if:*

(a) (non-idempotency) $\zeta \notin \{0, e_+, e_-, 1\}$*, i.e. ζ is not one of the four idempotent atoms;*

(b) (σ-fixedness) $\sigma(\zeta) = \zeta$*, where $\sigma : \mathbb{D} \to \mathbb{D}$ is the lobe-swap involution $\sigma(ae_+ + be_-) = be_+ + ae_-$;*

(c) (balance) $\zeta \in (0,1) \cdot 1 \subset \mathbb{D}$*, i.e. ζ is a scalar multiple of the unit 1 in the open unit interval (the non-degenerate σ-fixed locus);*

(d) (stability) *ζ is tail-stable: represented by an ω-germ in the primorial tower (Definition* **??***).*

Theorem 11.12 (Hinge-3 constant is the ω-atom **[τ-Effective]**)**.** *The master constant $\iota_\tau = 2/(\pi + e)$ of Hinge 3 is the unique ω-atom of $\mathbb{D}$ in the sense of Definition* **??**.

Proof. Existence and uniqueness of an element satisfying (a)–(d) is Theorem **??** of the present paper: ι_τ exists as the canonical balanced ω-germ, and no other element of $\mathbb{D}$ satisfies (a)–(d).

Identification with the Hinge-3 value: by the uniqueness corollary of [**?**], the scalar readout of the unique σ-fixed non-polar ω-approaching germ on $\mathbb{L}$ is $\iota_\tau = 2_\tau/(\pi_\tau + e_\tau) = 2/(\pi + e)$. Since the Hinge-4 balanced ω-germ is constructed by the same primorial refinement as the Hinge-3 crossing germ (both inherit refinement compatibility from Lemma **??**), the scalar readouts agree: $\iota_\tau^{\text{Hinge 4}} = \iota_\tau^{\text{Hinge 3}}$.

Finally, (a)–(d) match (a)–(c) of Definition **??**: non-idempotency is non-degeneracy, σ-fixedness is balance, tail-stability is the ω-germ condition. □

Corollary 11.13 ($\omega \leftrightarrow \iota_\tau$ in the atom dictionary)**.** *Under the τ-generator dictionary of §??, the τ-generator ω (Higgs, completion) is realised in $\mathbb{D}$ as the ω-atom $\iota_\tau = \mathrm{Mediator}(\{e_+, e_-\})$, the unique σ-fixed non-idempotent mediator of the two idempotent atoms.*

Proof. Immediate from Theorem **??** and the atom dictionary of §**??**. □

Remark 11.14 (Why ι_τ is not itself an idempotent)**.** $\iota_\tau^2 = \iota_\tau \cdot 1 \neq \iota_\tau$ unless $\iota_\tau \in \{0, 1\}$. Since $\iota_\tau \in (0,1)$, the scalar ι_τ is strictly non-idempotent; it lies in the open interior of the σ-fixed axis $\{a \cdot 1 : a \in \mathcal{R}_\partial\}$ of $\mathbb{D}$, with 0 and 1 as its two idempotent σ-fixed endpoints. This is why ι_τ is the *fifth* atom in the dictionary, and why the dictionary is "four atoms plus a completion": the four idempotent atoms $\{0, e_+, e_-, 1\}$ exhaust the σ-invariant idempotent spectrum, and ι_τ is the unique non-idempotent σ-fixed completion.

11.5 Consistency: the three lifts agree

We verify that the three lifts (1), (2), (3) of Theorem **??** are mutually consistent: evaluating the Hinge-1 coordinate chart on the Hinge-2-classified primes, along the primorial ladder, yields the Hinge-3 master constant in the refinement limit.

Proposition 11.15 (Triple consistency at the primorial limit **[τ-Effective]**)**.** *Let $M_k = \prod_{n \le k} p_n$ be the primorial ladder. For each k, let $X_k := M_k$ be the kth primorial. Then:*

(i) $\Phi_{\text{ch}}(X_k) = (p_k, 1, 1, M_{k-1})$*, so* $\mathrm{coord}_A(X_k) = p_k$, $\mathrm{coord}_B(X_k) = \mathrm{coord}_C(X_k) = 1$, $\mathrm{coord}_D(X_k) = M_{k-1}$ *(Hinge 1 [**?**] on primorials);*

(ii) $\tilde{\chi}(X_k) = \nu_B(M_k)\, e_+ + \nu_C(M_k)\, e_-$*, and by Hinge 2 [**?**], $\nu_B(M_k)/k \to 1/2$ and $\nu_C(M_k)/k \to 1/2$ as $k \to \infty$ (Hinge 2 densities);*

(iii) *the normalised primorial scalar readout ρ_k defined by [**?**] converges to ι_τ:* $\lim_{k\to\infty} \rho_k = \iota_\tau$ *(Hinge 3 refinement).*

Moreover, the three limits are coherent: the Hinge-3 refinement-limit ι_τ is the unique σ-fixed $\mathbb{D}$-scalar compatible with (i) the Hinge-1 "$C = 1$, $B = 1$" primorial specialisation (which forces the tower atom $T(A, B, C) = A$ to act as a pure prime factor) and (ii) the Hinge-2 density-$\frac{1}{2}$ balance between $\mathbb{P}_B$ and $\mathbb{P}_C$ (which forces $\mathrm{Tr}_-(\tilde{\chi}(M_k))/k \to 0$, i.e. σ-invariance at the refinement limit).

Lean-grade sketch. (i) By definition, M_k is squarefree with largest prime factor p_k; the ABCD algorithm peels p_k to height $C = 1$, exponent $B = 1$, leaving $D = M_{k-1}$. This is the primorial specialisation of [**?**].

(ii) Immediate from the density-$\frac{1}{2}$ statement in [**?**] and the squarefree factorisation $M_k = \prod_{n \le k} p_n$.

(iii) This is the primorial-limit theorem of [**?**], restated in the Hinge-4 notation.

Coherence: the σ-invariance condition on the refinement limit forces $\mathrm{Tr}\_(\tilde{\chi}(M_k)) \to 0$, which by Proposition **??**(iii) equals $\chi(M_k) = \nu_B(M_k) - \nu_C(M_k) \to 0$; this is precisely the Hinge-2 density-$\frac{1}{2}$ statement. Conversely, σ-invariance of the primorial ω-germ fixes the scalar readout to be a pure multiple of $1 = e_+ + e_-$, and the unique such balanced tail-stable non-idempotent scalar is ι_τ (Theorem **??**). The Hinge-1 primorial specialisation $\mathrm{coord}_B = \mathrm{coord}_C = 1$ guarantees the tower atom acts by pure prime multiplication, which is compatible with the additive structure of $\tilde{\chi}$ (Proposition **??**(i)). $\square$

11.6 A unified algebraic language for the hinge bundle

Theorem **??** induces a structural taxonomy of the three prior hinges in terms of three distinct $\mathbb{D}$-valued object classes:

Hinge	**$\mathbb{D}$-valued object class**	**Canonical object**
Hinge 1 (Hyperfact.)	$\mathbb{D}$-valued coordinate functions on $\mathcal{R}_\partial$	$\widehat{\Phi}_{\mathrm{ch}} : \mathcal{R}_\partial \to \mathbb{D}^4$
Hinge 2 (Prime Polarity)	$\mathbb{D}$-valued completely-additive characters on $(\mathbb{N}, \times)$	$\tilde{\chi} : (\mathbb{N}, \times) \to (\mathbb{D}, +)$
Hinge 3 (Master Constant)	Distinguished σ-fixed non-idempotent $\mathbb{D}$-scalars	$\iota_\tau \in \mathbb{D}$
Hinge 4 (this paper)	The algebra $\mathbb{D}$ itself, with atom dictionary	$\mathbb{D} \supset \{0, e_+, e_-, 1, \iota_\tau\}$

The table exhibits Hinge 4 as the *ambient* hinge: it provides the algebra $\mathbb{D}$ in which the other three hinges' central objects live. Hinges 1, 2, 3 are, respectively: (1) a chart, (2) a character, (3) a distinguished scalar — three natural object classes over any split-complex scalar algebra. The unification is not an analogy; it is an algebraic *containment*.

Remark 11.16 (Why this containment was not visible earlier)**.** Hinges 1, 2, 3 were originally presented in their own self-contained ambient categories: Φ_{ch} lives in $\mathbb{P} \times \mathbb{N}^3_{\geq 1}$, χ lives in the integer sign group $\{\pm 1\}$, and ι_τ lives in $\mathbb{R}$. The common scalar algebra $\mathbb{D}$ becomes visible only when (a) the target category is upgraded from $\mathbb{R}$ to the split-complex $\mathbb{D}$ (Hinge 4 §**??**), and (b) the domain is upgraded from $\mathbb{N}$ to the profinite $\mathcal{R}_\partial$ (Hinge 4 §**??**). These upgrades are exactly what Hinge 4 supplies. It is natural, in retrospect, that the bundle *has* a structural hinge: each prior hinge produces an object that needs a scalar algebra, and $\mathbb{D}$ is the unique such algebra compatible with all three.

Remark 11.17 (Compatibility with Books I–III)**.** The unified view is consistent with the Category-τ architecture of Books I–III. Specifically:

- Book I [**?**] Ch. 24 (greedy peel on τ-Idx) gives the Hinge-1 chart in Category τ; its lift to $\mathcal{R}_\partial$ via the Hyperfactorization isomorphism [**?**] is Proposition **??** of the present section.
- Book II [**?**] Ch. 47 (idempotent-completeness lemma) establishes the split-complex algebra H_τ^{cal} as the completion of the spectral character algebra; our $\mathbb{D}$ is the boundary-native constructive shadow of that completion.
- Book III [**?**] Ch. 5 (spectral tower on $\mathbb{L}$) supplies the refinement tower that produces ι_τ in the limit; our registry entries III.T81–III.T89 record the Lean-grade form of this correspondence.

In this sense, Hinge 4 is the standalone-paper projection of the Book I–II–III architecture onto a single scalar algebra.

11.7 Forward links: what Hinge 4 enables

The unified $\mathbb{D}$-based view opens three concrete forward directions, each of which is the subject of a candidate Hinge 5–Hinge 6 paper or a future-book application.

Hinge 5 candidate: $\mathbb{D}$-holomorphy and the wave equation.. The split-complex unit j satisfies $j^2 = +1$, so the $\mathbb{D}$-analogue of the Cauchy–Riemann equations is the *hyperbolic* wave equation $\partial_t^2 f = \partial_x^2 f$, not the elliptic Laplace equation $\partial_t^2 f + \partial_x^2 f = 0$. This is the algebraic face of the Elliptic Exclusion principle of §**??**: boundary-native holomorphy is Lorentzian, not Euclidean. The candidate Hinge 5 paper will formalise $\mathbb{D}$-holomorphic function theory ($\mathbb{D}$-analytic maps, $\mathbb{D}$-Cauchy integrals, $\mathbb{D}$-residues) and derive the boundary wave equation as its canonical PDE. A starting-point monograph is [**?**].

Hinge 6 candidate: the τ-Topos over $\mathbb{D}$.. The natural scalar base for sheafified boundary assignments is $\mathbb{D}$: each boundary stalk at a lemniscate point is a $\mathbb{D}$-module, the structure sheaf is a sheaf of $\mathbb{D}$-algebras, and the global sections functor is the Hinge-4 character-realisation functor. A candidate Hinge 6 paper will formalise this topos and show it recovers the ω-category structure of Book II [**?**] via its internal logic.

Book IV applications: sector coupling via idempotent projections.. The physics applications in Book IV [**?**] (electroweak, Weinberg NLO, W-sum identities) use the force-generator identification π = weak, γ = EM, η = strong, α = gravity, ω = Higgs (locked 2026-02-16). The present $\mathbb{D}$-integration gives a kernel-level algebraic witness for this mapping: each force corresponds to an idempotent or mediator atom of $\mathbb{D}$, and sector coupling is a $\mathbb{D}$-module morphism. In particular:

- $\kappa_D = 1 - \iota_\tau$ (diagonal closure coupling) is the complement of the ω-atom ι_τ within the unit atom 1;
- $\kappa_\omega = \iota_\tau/(1+\iota_\tau)$ is the ω-normalised σ-fixed coupling, the natural renormalisation of ι_τ under idempotent enlargement $1 \mapsto 1 + \iota_\tau$;
- the W-sum identities $W_3(4) = 5$, $W_3(3) = 17$, etc., are Archimedean readouts of the ABCD refinement towers evaluated on the generators in their Hinge-4 $\mathbb{D}$-sector.

These identifications are the algebraic backbone of Book IV's physics ledger; the present paper provides their canonical common home.

Books V–VII forward links.. Books V–VII [**?**, **?**, **?**] (macrocosm, life, metaphysics) all depend on the σ-fixed balancing property of ι_τ as their scalar-calibration axis. The Hinge-4 framing promotes this from a numerical coincidence to an algebraic theorem: ι_τ is the unique ω-atom of the unique split-complex boundary algebra $\mathbb{D}$; hence any theory whose scalar layer is σ-fixed and boundary-native must calibrate to ι_τ. This is the structural content underlying the Books V–VII calibration cascades.

Summary of the hinge bundle. The four hinges together form a complete algebraic-structural package:

Hinge 1	$X = (A \uparrow\uparrow C)^B \cdot D$ — the integer chart into $\mathbb{D}^4$.
Hinge 2	$\chi = \mathrm{Leg}(2/p)$ — the prime character into $\mathbb{D}$-atoms.
Hinge 3	$\iota_\tau = 2/(\pi + e)$ — the σ-fixed scalar in $\mathbb{D}$.
Hinge 4	$\mathbb{D} = \mathcal{R}_\partial[j]/(j^2 - 1)$ — the unique algebra hosting all three.

What the bundle says, in one line: *the split-complex boundary algebra $\mathbb{D}$ is the canonical common home of the Panta Rhei research programme's arithmetic, spectral, and scalar hinges.* All four hinges are theorems about $\mathbb{D}$; Hinge 4 is the theorem naming $\mathbb{D}$ itself.

12. LEAN ROADMAP AND ARTIFACTS

- Primorial ladder `M : Idx -> TauInt` and its reduction maps.
- Tower type `Tower` with compatibility predicate `Compat`.
- Boundary ring `Rbd` as stabilized compatible towers.
- CRT isomorphisms at finite stages plus inverse limit coherence lemmas.
- Lift operator `Lift` (primitive recursion) plus factorization lemma.
- Split-complex scalars `D = Rbd × Rbd` plus idempotent projectors.
- Mediator `iota` as unique minimizer or fixed point.

13. REGISTRY IDENTIFIERS AND LEAN CONNECTION

Remark 13.1 (Registry IDs **[τ-Effective]**)**.** Boundary ring $\mathcal{R}_\partial = \varprojlim_k \mathbb{Z}/M_k\mathbb{Z}$ at the lemniscate: III.T81 (**[Established]** profinite limit construction); split-complex embedding and canonical idempotents $1 = e_+ + e_-$: III.T84 (**[τ-Effective]** connection to τ-boundary); idempotent decomposition of character algebra: III.T85 (**[τ-Effective]**); crossing mediator $\iota_\tau = 2/(\pi + e)$: III.T86 (**[τ-Effective]**). Formalized in `TauLib.BookIII.BoundaryRing`.

14. CONCLUSION AND FORWARD LINKS

This paper has established the split-complex boundary algebra $\mathbb{D}$ as the canonical scalar algebra of Category τ, unique among commutative $\mathcal{R}_\partial$-algebras under the four τ-kernel structural constraints (Theorem **??**), and has shown that $\mathbb{D}$ is the unifying algebraic home of the other three hinge papers of the bundle (Theorem **??**). The four-atom spectral dictionary (Theorem **??**) gives the explicit bridge from $\mathbb{D}$ to the 2nd-Edition force-mapping generators, and the elliptic complex exclusion theorem (Theorem **??**) rules out the Gaussian alternative definitively.

Forward directions opened by Hinge 4:

- **Hinge 5 candidate: $\mathbb{D}$-holomorphy.** The split-complex unit $j^2 = +1$ forces the $\mathbb{D}$-Cauchy–Riemann equations to decouple into the *wave equation* $\partial_t^2 f = \partial_x^2 f$ (rather than the elliptic Laplace equation of orthodox complex holomorphy). A companion paper formalising $\mathbb{D}$-analytic function theory, $\mathbb{D}$-Cauchy integrals, and the boundary wave equation is a natural follow-up; starting-point monograph [**?**].
- **Hinge 6 candidate: the τ-Topos over $\mathbb{D}$.** Each boundary stalk at a lemniscate point is a $\mathbb{D}$-module; the structure sheaf is a sheaf of $\mathbb{D}$-algebras. A companion paper formalising this topos and recovering the ω-category structure of Book II [**?**] via its internal logic is a natural follow-up.
- **Physics layer (Book IV [?]):** sector-level fixed-point readouts (Weinberg angle, fine-structure constant, strong coupling) are ι_τ-calibrated projections of specific boundary characters via Corollary **??**. The identifications $\kappa_D = 1 - \iota_\tau$ and $\kappa_\omega = \iota_\tau/(1 + \iota_\tau)$ are algebraic corollaries of the four-atom dictionary.
- **Books V–VII [?, ?, ?]:** the σ-fixed balancing property of ι_τ as scalar-calibration axis is elevated here from numerical coincidence to algebraic theorem: ι_τ is the unique ω-atom of $\mathbb{D}$; hence any theory whose scalar layer is σ-fixed and boundary-native must calibrate to ι_τ.

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Data and code availability

The source repository for the paper bundle is at `https://panta-rhei.site/papers/boundary-algebra`. Planned Lean 4 artefacts for the uniqueness, elliptic-exclusion, four-atom dictionary, and hinge-integration theorems will appear in `TauLib.BookIII.BoundaryAlgebra` (see the Lean roadmap in §**??**).