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The Master Constant ι_τ

A structural derivation

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ABSTRACT

We give a fully structural derivation of the master constant ι_τ of Category τ, identifying it as the canonical scalar readout of the unique σ-fixed crossing-point ω-germ on the lemniscate boundary and proving the coupling identity

$$\boxed{\iota_\tau \;=\; \frac{2_\tau}{\pi_\tau + e_\tau}}$$

as a normalisation theorem between three independently earned τ-native invariants: (i) the dyadic refinement constant 2_τ (radial ultrametric branching), (ii) the Euclidean incidence invariant π_τ (circle-vs.-radius refinement growth), (iii) the τ-exponential invariant e_τ (canonical σ-equivariant holomorphic boundary transformer).

The derivation has three substantive parts. *First*, we construct each canonical invariant as an ω-germ (an inverse-limit boundary object) inside τ, without importing uncountable completions. In particular, π_τ and e_τ arise as refinement-invariants of the lemniscate-boundary kernel, not as rescalings of orthodox π, e. The crossing-point defect germ is built by an explicit five-step construction program (maximal γ/η torus; realised pairing; defect; refinement compatibility; σ-invariance), and its convergence is controlled along primorial refinement radii. *Second*, we establish the *crossing-point uniqueness principle* via a two-pronged argument: (a) a *non-polarity* half, where the lobe-swap involution Swap_n on the depth-n polarity lattice is shown to preserve transport closure, fusion admissibility, associativity, and anchor rigidity — four lobe-invariance lemmas that force every σ-fixed non-polar germ to coincide with the crossing-point germ $G_\times[\omega]$; and (b) an *ω-approach* half, where a meta-witness depth function together with a refinement-pressure lemma shows that unbounded refinement in the boundary filtration forces any candidate mediator to be ω-approaching. The intersection of the two classes (non-polar $\cap$ ω-approaching) is a singleton, namely ι_τ. Consequently ι_τ is invariant under the entire σ-equivariant holomorphic endomorphism monoid $\mathrm{HolEnd}^\sigma_\tau(\omega)$ — a *universal fixed scalar*. *Third*, we prove the coupling identity by finite-stage normalisation: at each depth n we exhibit refinement-compatible approximants satisfying $\iota_\tau^{(n)} = 2_\tau/(\pi_\tau^{(n)} + e_\tau^{(n)})$, and passage to the ω-limit extracts the coupling identity as a statement in the τ boundary scalar algebra. The numerical projection under the canonical scalar-readout functor recovers $\iota_\tau \approx 0.341304238875$, matching $2/(\pi + e)$ when the orthodox values of π and e are substituted. The coupling identity is proved via *normalisation-by-uniqueness*: given σ-equivariance of the boundary scalar algebra and the universality of the crossing germ under the σ-equivariant holomorphic endomorphism monoid $\mathrm{HolEnd}^\sigma_\tau(\omega)$, the orthogonal-idempotent decomposition of Book II Chapter 47 [4] forces the unique refinement-compatible combination of π_τ and e_τ to be the additive trace $\pi_\tau + e_\tau$, and the unique scalar satisfying this normalisation against the dyadic clock 2_τ is ι_τ. The coupling identity takes the idempotent form $\iota_\tau \cdot \mathrm{Tr}_+(w_\omega) = 2_\tau$ with $w_\omega = \pi_\tau \mathbf{e}_+ + e_\tau \mathbf{e}_- \in \mathbb{D} \otimes \mathbb{R}_\tau$. The prime-polarity character $\chi : (\mathbb{N}, \times) \to (\mathbb{Z}, +)$ ($p_B \mapsto +1, p_C \mapsto -1, 1 \mapsto 0$, matching Book II Chapter 47's $B \leftrightarrow \mathbf{e}_+, C \leftrightarrow \mathbf{e}_-$) lifts to a completely additive monoid homomorphism $\tilde{\chi} : (\mathbb{N}, \times) \to (\mathbb{D}, +)$, establishing the algebraic bridge to the companion Prime Polarity theorem (a Dirichlet density on primes) and the identification of the B/C channel split with the final 2nd-Ed generator split γ/η. The boundary–interior identification of e_τ with Book II's radial ν-iterator eigenvalue (v2.2–v2.6 OQ5) is *closed* unconditionally (Theorem 7.22) via Book II's Mutual Determination (II.T27) in its (G)↔(R) form, using the identification $\sigma =$ bipolar swap and a Yoneda-style uniqueness argument (Lemma 7.21). The ramification triviality (v2.7 OQ4) is closed as Proposition 7.11: the prime-polarity lift $\tilde{\chi}$ vanishes on all powers of the ramified prime $p = 2$, identifying the ramified prime with the dyadic clock 2_τ. The polarised-germ structure (v2.7 OQ2) is closed in its structural–geometric part: Theorem 8.5 establishes canonical existence and uniqueness of the B- and C-polarised ω-germs via lobe-restricted L1–L4 invariance, and Theorem 8.8 identifies both polarised scalar readouts as $\kappa_D = 1 - \iota_\tau$ (common, forced by σ-equivariance of the readout functor). Corollary 8.9 supplements this with the algebraic Möbius companion $\kappa_\omega = \iota_\tau/(1 + \iota_\tau)$, closing the triad $\{\iota_\tau, \kappa_D, \kappa_\omega\}$ algebraically; the structural identification of κ_ω with the ω-generator / Higgs sector is a Book IV forward reference. All v1–v2 structural obstructions to the coupling identity derivation are closed.

This paper provides a *second, structurally independent proof* of the 2nd-edition Book II Theorem II.T25 (ch. 28, assembly of $\iota_\tau = 2/(\pi + e)$). Book II proves II.T25 by *calibration*: π earned via solenoidal-circle geometry (II.T22), e earned via ν-iterator eigenvalue (II.T23), then the ratio observed. The present paper proves the same numerical identity by a different route — σ-equivariance plus crossing-germ uniqueness plus primorial combinatorial linearity — and thereby supersedes the 2nd-edition Book I Chapter 41 treatment of ι_τ, which defined the constant by fiat and claimed (incorrectly, as shown in the companion Prime Polarity paper [10]) that ι_τ equals the B/C prime density ratio. A companion errata entry (ERRATUM-004) records the correction. The numerical agreement between the two independent proofs is itself a non-trivial consistency check across Books I, II, and III.

Keywords Master constant ; ω-germs ; Lemniscate boundary ; Crossing-point germ ; σ-equivariance ; τ-exponential ; Coupling identity ; Transcendentals in τ ; Category τ ; Lean 4 formalisation

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1. INTRODUCTION

1.1 Position in the hinge-paper bundle

This paper is **Hinge 3** of the eight-paper Panta Rhei foundational bundle accompanying the 2nd Edition of the series [3, 4, 5]. The bundle consists of seven technical hinges (H1–H7) plus a foundational-anchor paper (H8); in the recommended reading order they are:

Hinge 1: *Hyperfactorization* [2] — unique tower-atom decomposition $X = (A \uparrow\uparrow C)^B \cdot D$ in ZFC and in Category τ, with an Isomorphism Theorem; supplies the I.T04 coordinate functions used downstream.

Hinge 2: *Prime Polarity* [10] — classifies the rational primes into B/C channels via the Legendre symbol $(2/p)$, proved pointwise equivalent to a τ-internal CRT-idempotent-plus-split-complex classifier ($\text{Label}_\infty \equiv \text{Pol}$).

Hinge 3: *Master Constant ι_τ (this paper)* — structural derivation of $\iota_\tau = 2/(\pi + e) \approx 0.341304$ as the canonical scalar readout of the unique σ-fixed crossing-point ω-germ on the lemniscate, with the split-complex idempotent lift $\tilde{\chi}$ of Prime Polarity's character χ providing the algebraic bridge between Hinges 2 and 3.

Hinge 4: *The Split-Complex Boundary Algebra* [11] — proves that the split-complex algebra $\mathbb{D} = \mathcal{R}_\partial[j]/(j^2 - 1)$ is the unique τ-admissible scalar algebra under four kernel constraints, rules out the elliptic $\mathbb{Z}[i]$ alternative via a no-go theorem, establishes the four-atom spectral dictionary linking $\mathbb{D}$ to the τ-generators, and exhibits $\mathbb{D}$ as the common algebraic home of all three prior hinges (including the $\tilde{\chi}$ of the present paper).

Hinge 5: *τ-Holomorphy on the Boundary Algebra* [12] — installs τ-holomorphy as the ontological primary; earned categorical machine; wave-equation Cauchy–Riemann; pre-Yoneda collapse.

Hinge 6: *The τ-Topos and Its Four-Valued Internal Logic* [14] — builds the τ-topos $\mathbf{Cat}_\tau$ with subobject classifier $\Omega_\tau = B_\sigma$ and paraconsistent Belnap–Dunn internal logic resolving semantic circularity.

Hinge 7: *Address Resolution, Not Calculation* [1] — NF confluence (Church–Rosser for the τ-kernel), genealogical DAG, Cayley word metric, ontic ultrametric; arithmetic in Category τ is address-resolution, not equational calculation.

Hinge 8: *The τ-Kernel as Foundational Architecture* [13] — foundational-anchor paper (also readable as an entry point): ontic identity invariance, diagonal–linear correspondence, $*$-autonomous placement; names what the seven technical hinges collectively earn.

Each paper is standalone-readable. Hinge 8 may be read first (as an entry) or last (as a capstone); the other seven may be read in the dependency order above. The present paper (Hinge 3) assembles the crossing-point germ and identifies its scalar readout with ι_τ, using tools from Hinges 1 and 2 and preparing the ground for Hinges 4–8.

1.2 Motivation and scope

The master constant ι_τ is a load-bearing invariant of the *Panta Rhei* research programme [3, 4, 5]: it appears in the coupling of arithmetic, spectral, and physical layers and is asserted throughout the series as the unique distinguished non-generator invariant of Category τ. The numerical value $\iota_\tau = 2/(\pi + e) \approx 0.341304238875$ appears repeatedly.

Two concerns motivate the present paper.

First, the 2nd-edition Book I Chapter 41 treatment postulates $\iota_\tau := 2/(\pi + e)$ by fiat and identifies it with the asymptotic B-class prime density (which the 1st-Ed loosely called "the B/C ratio R_B"; we reserve ρ_B below for the corrected density). The companion Prime Polarity hinge paper [10] establishes rigorously that $\rho_B := \lim_{N\to\infty} |\{p \le N : p \in \mathbb{P}_B\}|/|\{p \le N\}| = 1/2$ (natural density, via Dirichlet's theorem on primes in arithmetic progressions), which is *not* $2/(\pi + e)$. Hence the Chapter 41 identification is incorrect and the constant itself requires a different structural foundation.

Second, the derivation must meet the "crystal-clear not-numerology" bar: π_τ and e_τ must arise as τ-native invariants inside the kernel, not as orthodox constants re-imported from ZFC and rescaled. The current 2nd-edition ch41 defines them as $\pi_\tau := \pi \cdot \iota_\tau$ and $e_\tau := e \cdot \iota_\tau$, which fails this bar.

This paper establishes the correct derivation: ι_τ as the canonical scalar readout of the unique σ-fixed crossing-point ω-germ, and the coupling identity as a normalisation theorem among three independently earned ω-germ invariants. The derivation is entirely internal to τ; orthodox π, e enter only at the numerical-projection step via the canonical scalar-readout functor.

1.3 The structure of the derivation

The paper develops a single proof chain:

(1) §2 introduces ω-germs as inverse-limit boundary objects, the τ-native replacement for "limit points in an uncountable continuum".

(2) §3 defines the three canonical invariants: 2_τ (dyadic refinement branching), π_τ (circle/radius refinement growth), e_τ (scalar readout of the unique σ-equivariant minimally-advancing holomorphic boundary transformer; the τ-exponential $\mathcal{E}$).

(3) §4 constructs the crossing-point defect ω-germ Δ_ω from the defect inverse system $\{T_n \setminus R_n\}$ of torus-projection failures, organised into an explicit *five-step construction program* (maximal γ/η torus, realised pairing, defect, refinement compatibility, σ-invariance) and shown to converge along the *primorial refinement filtration*.

(4) §5 proves the crossing-point uniqueness principle in two halves: §5.2 (non-polarity) — any σ-fixed non-polar germ is fixed by the lobe-swap involution Swap_n at every depth, whose four lobe-invariance lemmas (transport closure, fusion admissibility, associativity, anchor rigidity) pin it to $G_\times[\omega]$; and §5.3 (ω-approach) — a meta-witness depth function and a refinement-pressure lemma show that no non-crossing candidate can survive unbounded refinement. The intersection (§5.4) is a singleton; ι_τ is therefore invariant under $\mathrm{HolEnd}^\sigma_\tau(\omega)$, a universal fixed scalar.

(5) §6 proves the coupling identity $\iota_\tau = 2_\tau/(\pi_\tau + e_\tau)$ as a normalisation theorem: at each finite depth n, $\iota_\tau^{(n)} = 2_\tau^{(n)}/(\pi_\tau^{(n)} + e_\tau^{(n)})$ by construction, and refinement compatibility extracts the inverse-limit identity.

(6) §7 establishes the numerical isomorphism with orthodox π, e: the canonical scalar-readout functor sends $\pi_\tau \mapsto \pi$, $e_\tau \mapsto e$, $2_\tau \mapsto 2$, and hence $\iota_\tau \mapsto 2/(\pi + e) \approx 0.341304$. The readout is further refined into a *split-complex idempotent lift* $\tilde{\chi}$ of the Dirichlet character χ of prime polarity, giving an algebraic bridge to the Prime Polarity theorem.

(7) §8 treats consequences: reconciliation with the falsified $R_B = \iota_\tau$ claim, the identification $B \leftrightarrow \gamma$, $C \leftrightarrow \eta$ under the locked 2nd-Ed force mapping, the Legendre $(2/p)$ prime split as τ-native γ/η split (via the second supplementary law of QR, aligning with the companion Prime Polarity paper's Isomorphism Theorem), and the $\mathrm{Enrich}^4(\tau) = \mathrm{Enrich}^3(\tau)$ *saturation theorem* that closes the self-enrichment tower at level 3.

1.4 Relation to the 2nd-Edition Book II manuscript

The coupling identity $\iota_\tau = 2/(\pi + e)$ is already stated in the 2nd-Edition Book II manuscript as Theorem II.T25 (*iota-tau confirmed*, Chapter 28 [4]), with the supporting infrastructure distributed across Part IV (Tarski primitives: II.T15–T18), Part V (earned constants: II.T22 for π, II.T23 for e), and Part IX (idempotent decomposition: II.P14). The role of the present paper is therefore *not* to establish the numerical identity, which is a theorem of the monograph, but to supply the *structural backbone* under which II.T25 is a corollary rather than an assertion:

(1) the *crossing-point defect ω-germ* Δ_ω as the inverse-limit object whose scalar readout is ι_τ (§4);

(2) the *crossing-point uniqueness principle* (§5) that pins ι_τ as a universal fixed scalar, not merely a numerical coefficient;

(3) the *finite-stage normalisation identity* $\iota_\tau^{(n)} \cdot (\pi_\tau^{(n)} + e_\tau^{(n)}) = 2_\tau + \varepsilon_n$ (§6, Lemma 6.3), proved via a two-step argument combining (2a) primorial-stabilised combinatorial linearity of the defect-torus counting and (2b) σ-invariance selecting the additive trace Tr_+ uniquely among $\mathbb{Z}$-linear functionals on the idempotent-decomposed boundary scalar algebra.

In this sense, the present paper provides a *second, structurally independent proof* of Book II Theorem II.T25. Book II's original proof is a calibration argument (earned π via II.T22 plus earned e via II.T23, then assembled). The present proof uses a completely different architecture (σ-equivariance, crossing-germ uniqueness, primorial combinatorial linearity) and arrives at the same numerical identity, which is a non-trivial consistency check on the τ-framework rather than a logical dependency.

Precise sense of independence. The Book II calibration proof of II.T25 (Chapter 28) proceeds by *assembly*: it combines the three earned-constant outputs of Chapters 25–26 (II.T22 for π, II.T23 for e) and Chapter 28's own primorial-sieve factorisation (II.P06 + II.D34) into a single calibration identity without an intermediate structural pivot. The present proof does *not* use II.T22 or II.T23 at all as earned-constant statements — π_τ and e_τ enter here only as scalar readouts of ω-germs constructed locally in §3, whose identification with orthodox π and e is performed by the canonical scalar-readout functor in §7, not inherited from Chapters 25–26. The architecture instead pivots on three Book II tools used elsewhere but not in Chapter 28: the BndLift norm bound of II.T26 (Ch. 30), the Mutual-Determination bijection of II.T27 (Ch. 31), and the idempotent character decomposition II.P14 (Ch. 47). The shared infrastructure with Chapter 28 is exactly II.P06 + II.D34 (primorial-sieve factorisation), imported as an external lemma in Lemma 6.3 Step 2a; that import is scoped and labelled honestly as an external black-box in Remark 6.4. The resulting independence is structural, not literal: the two proofs share one combinatorial pillar but assemble it through disjoint architectural routes. The companion errata entry ERRATUM-004 records the consequent correction to Book I Chapter 41, which defined ι_τ by fiat and wrongly identified it with the B/C prime density ratio.

1.5 What this paper is not

This paper is scoped to the structural derivation and its integration with the 2nd-Edition Book II manuscript. We do *not* here:

- prove the full universality of ι_τ as generator of the τ-fixed subfield of physical constants (this is Book IV [6] territory);
- treat the p-adic / profinite aspects of ω-germs in full analytic rigour (these belong to Book III [5]);
- undertake the Lean formalisation (see Appendix A for the proof-chain sketch).

1.6 Dependencies and conventions

For the reader's convenience, we list the main Book II and Book III results imported throughout this paper, with one-sentence paraphrases. Full statements are in the cited chapters of the monograph [4, 5]; the present paper is not self-contained with respect to these imports and assumes them as given.

Book I generator ladder (I.D04):
The four-fold generator decomposition $\rho \to \mu \to \nu \to \theta$ of the kernel refinement map; ν is the multiplicative step governing D-channel growth.

Book II Chapter 17, Theorem II.T13:
Torus degeneration $T^2 \to \mathbb{L}$ via the pinch map.

Book II Chapter 19, Definition II.D20 + Theorem II.T16:
Tarski-primitive congruence predicate derived from the ultrametric.

Book II Chapter 23–24, Definitions II.D24, II.D26, II.D27, Theorem II.T21:
Line ℓ_D and solenoidal-circle S^X constructions; ω-germs as inverse-limit profinite objects.

Book II Chapter 25, Theorem II.T22:
Three perspectives on π: topological, geometric (Archimedes polygon), spectral, all equal.

Book II Chapter 26, Definition II.D30 + Theorem II.T23:
e earned as ν-iterator eigenvalue $e_\nu = \lim_k (1 + 1/p_{k+1})^{p_{k+1}}$, with transcendence.

Book II Chapter 28, Theorem II.T25 + Proposition II.P06 + Definition II.D34:
$\iota_\tau = 2/(\pi + e)$ calibration identity; refinement-resolution duality $\log_2 P_k \sim k \ln k / \ln 2$; Archimedean bridge $\iota_\tau \cdot 2^{-\delta}$.

Book II Chapter 30, Definition II.D36 + Theorem II.T26:
BndLift operator with norm bound $(1 + 2\iota_\tau/p_{n+1})$ per stage.

Book II Chapter 31, Theorem II.T27:
Mutual Determination: canonical five-way equivalence (G)↔(S)↔(R)↔(C)↔(H) preserving tower grading, finite spectral support, and bipolar decomposition.

Book II Chapter 32, Definition II.D37:
Evolution operator $\mathcal{E}_{n \to m} := \mathrm{BndLift}_{m-1} \circ \cdots \circ \mathrm{BndLift}_n$.

Book II Chapter 47, Definition II.D59 + Proposition II.P14:
Idempotent-supported character decomposition $\chi = \mathbf{e}_+ \chi_+ + \mathbf{e}_- \chi_-$; product isomorphism $A_{\mathrm{spec}}(\mathbb{L}) \cong A_\tau^{(B)} \times A_\tau^{(C)}$.

Book II Chapter 51, Theorem II.T40 with property (e):
Central Theorem canonical isomorphism $\mathcal{O}(\tau^3) \cong A_{\mathrm{spec}}(\mathbb{L})$ with ι_τ-calibration preservation of spectral-coefficient numerical values.

Book III Chapter 5:
Commutativity of Read with finite arithmetic in the boundary scalar algebra.

Throughout, the force mapping is the final 2nd-Ed locked convention (2026-02-16): $\alpha =$ gravity, $\pi =$ weak, $\gamma =$ EM, $\eta =$ strong, $\omega =$ Higgs; lemniscate lobes $B \leftrightarrow \gamma$, $C \leftrightarrow \eta$; idempotent correspondence $B \leftrightarrow \mathbf{e}_+$, $C \leftrightarrow \mathbf{e}_-$ (Book II Ch. 47 convention).

1.7 Revision history

This paper underwent twelve internal revision rounds (v1 through v3.3) with multiple peer-panel simulation rounds addressing structural rigour, scalar-identification consistency, and framing. Two rounds are especially worth surfacing explicitly. v3.0 corrected a stale numerical value for ι_τ (a 1st-Edition residue) and replaced it with the current $\iota_\tau = 2/(\pi + e) \approx 0.341304238875\ldots$ throughout. v3.1 had drafted a quantitative convergence rate for Theorem 4.8 by telescoping the Book II Ch. 30 per-stage BndLift bound $(1 + 2\iota_\tau/p_{n+1})$ into a Mertens-type partial sum; the final red-team pass caught that the telescoped sum $\sum_m 1/p_m$ in fact *diverges* by Mertens's theorem rather than converging as that draft assumed. v3.2 withdrew the flawed rate argument and replaced it with a purely qualitative Cauchy proof via inverse-limit compatibility; v3.3 completed the downstream cleanup by adding an explicit open-question entry (OQ4, §8.7) for the deferred rate, aligning scope language in the Open-Questions lead-in and the Conclusion, and splitting per-book citations (Books IV–VII) out of the generic PR-III bibliography entry. Over the course of all revisions, five originally-open structural questions were closed at the structural level: (i) primorial convergence (Theorem 4.8; qualitative Cauchy via inverse-limit compatibility, quantitative rate deferred as OQ4), (ii) angular–radial attribution resolved via the $\sigma =$ bipolar swap identification, (iii) boundary–interior identification $e_\tau = e_\nu$ (Theorem 7.22, Lemma 7.21), (iv) ramification triviality $\delta_{\text{ram}} = 0$ (Proposition 7.11 supported by Lemma 7.10), and (v) the polarised universal property plus common polarised readout (Theorems 8.5, 8.8). The self-enrichment saturation $\text{Enrich}^4(\tau) = \text{Enrich}^3(\tau)$ is stated as Conjecture 8.2 conditional on Book VII infrastructure. Corollary 8.9 closes the algebraic triad with the Möbius companion κ_ω. Four questions remain open as independent research directions (§8.7), none of them structural obstructions to the coupling identity derivation.

2. ω-GERMS: INVERSE-LIMIT BOUNDARY OBJECTS IN τ

2.1 The orthodox pipeline and its cost

In orthodox foundations, the standard number-theoretic pipeline is $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$, with π and e obtained as limits or analytic invariants inside $\mathbb{R}$ or $\mathbb{C}$. The cost of this pipeline is that it builds an *uncountable* completion $\mathbb{R}$ just to obtain a small, countable collection of named transcendentals; global gluing and analytic coherence must then be recovered *after* this completion.

In τ, the direction is reversed: global gluing is built at the kernel level, and constants arise as *boundary invariants* of refinement systems on ω-boundary objects. We briefly make this precise; full development is in Book I Part VII–IX [3].

2.2 Presentations and ω-germs

Let $\mathcal{B}_n$ denote the canonical depth-n clopen cylinder base on the τ-interior point object Pt (the latter itself being an ω-readout of the kernel, cf. Book I ch. 15 [3]). A *presented object* is a finite-stage refinement tower

$$X[1] \xleftarrow{r_{2\to1}} X[2] \xleftarrow{r_{3\to2}} X[3] \leftarrow \cdots, \tag{1}$$

where each $X[n]$ is τ-finite (encodable by τ-indices) and each $r_{n+1\to n}$ is a canonical refinement projection.

Definition 2.1 (ω-germ). *An* ω-germ *on the presentation* (1) *is the inverse-limit object*

$$X[\omega] \;:=\; \varprojlim_n X[n] \tag{2}$$

equipped with its canonical projection system. Equivalently, an ω-thread is a compatible family $(x_n)_{n\in\mathbb{N}_{\geq 1}}$ with $x_n \in X[n]$ and $r_{n+1\to n}(x_{n+1}) = x_n$.

Remark 2.2 (Countability and coherence). Each $X[n]$ is τ-finite, so the inverse system is internally countable; yet $X[\omega]$ is a genuine infinitary object with ω-coherent threads. This is the τ-replacement for uncountable limit points: countable in cardinality, infinitary in structure. The archetype is the solenoidal circle $S^X = \varprojlim_k C_k^X$ of Book II Chapter 24 [4] (Theorem II.T21), whose continuous surjection $\text{den}^X : S^X \to S^1$ onto the classical circle is the canonical model for how an ω-germ's Archimedean shadow recovers an orthodox continuum object.

2.3 The lemniscate boundary and the polarity involution

Remark 2.3 (Generator naming: $\pi'/\pi'' \to \gamma/\eta$). In the 1st-edition manuscripts and in earlier internal working sessions, the two non-Weak, non-Gravity, non-Higgs generators of Category τ were named π' and π''. In the final 2nd-edition manuscript (locked 2026-02-16) they were renamed to γ and η, so that the five generators read $(\alpha, \pi, \gamma, \eta, \omega)$ with the force mapping α=gravity, π=weak, γ=EM, η=strong, ω=Higgs. Throughout this paper, and in any prescribed replacement text for Book I Chapter 41, we use the final names γ and η. The two lemniscate lobes B and C below are identified, under this mapping, with the γ- and η-generator channels respectively (§8.2); older drafts and internal notes that write π' and π'' should be read under the substitution $\pi' \mapsto \gamma, \pi'' \mapsto \eta$.

The lemniscate $\mathbb{L} = S^1 \vee S^1$ is the canonical τ-boundary object (Book I ch. 40 [3]), with two lobes denoted B and C and crossing point $\omega_\times$. The *polarity involution*

$$\sigma : \mathbb{L} \to \mathbb{L} \tag{3}$$

exchanges the two lobes, fixing exactly the crossing point $\omega_\times$. On the boundary scalar algebra $H_\tau(\omega)$, σ extends to an algebra involution with $\sigma^2 = \mathrm{id}$.

Definition 2.4 (Polarised vs. unpolarised germ). *An ω-germ G on $\mathbb{L}$ is B-*polarised *if there exists a depth n_0 beyond which every compatible thread lies entirely in the B-lobe (and dually for C-polarised). G is* polarised *if it is B- or C-polarised. G is* unpolarised *if it is not polarised: at every depth $n \geq 1$ there exist threads with support in both lobes.*

Definition 2.5 (σ-fixed germ). *An ω-germ G is σ-*fixed *if $\sigma(G) = G$ as an inverse-limit object, i.e. the thread family is preserved under σ at every depth.*

3. THREE CANONICAL τ-INVARIANTS

We now define the three canonical ω-germ invariants that combine into ι_τ.

3.1 The dyadic refinement constant 2_τ

Definition 3.1 (Dyadic refinement constant 2_τ). *The dyadic refinement constant 2_τ is the canonical branching factor of the ultrametric cylinder base $\mathcal{B}_n$ at each refinement step. Concretely, $|\mathcal{B}_{n+1}|/|\mathcal{B}_n| = 2$ for every $n \geq 1$: each depth-n cylinder splits into exactly two depth-$(n+1)$ cylinders under the canonical refinement projection $r_{n+1\to n}$.*

2_τ is not imported as an external integer; it is the *structural fact* that the smallest non-trivial coherence-compatible branching of clopen cylinders is binary. Its scalar readout under the canonical readout functor is the integer 2.

3.2 The circle-refinement invariant π_τ

Definition 3.2 (τ-circle presentation). *Fix $O, R \in \mathrm{Pt}$. For each depth $n \geq 1$, define the τ-finite list*

$$C_{O,R}[n] \;:=\; \{\, U \in \mathcal{B}_n \;:\; \mathrm{Rep}(U) \textit{ satisfies } \|OX\| \equiv \|OR\| \pmod{2^{-n}} \,\} \tag{4}$$

*of depth-n cylinders whose canonical representatives satisfy the congruence predicate "distance from O equals distance from O to R" (in the sense of the Tarski-primitive congruence predicate derived from the ultrametric in Book II Chapter 19 [4], Definition II.D20 and Theorem II.T16). The ω-*circle *is the ω-germ*

$$C_{O,R}[\omega] \;:=\; \varprojlim_n C_{O,R}[n]. \tag{5}$$

Equivalently, this is the natural-number restriction of the solenoidal circle $S^X := \varprojlim_k C_k^X$ *of Book II Chapter 24 [4], Definition II.D26, with the geometric-topological unification of Definition II.D27 showing that the geometric and topological circles coincide as Archimedean shadows of the same profinite limit (Theorem II.T21).*

Definition 3.3 (Circle-refinement invariant π_τ). *The invariant π_τ is the ω-germ class of the* refinement growth ratio *between the circle presentation $C_{O,R}[n]$ and the radius-segment tower $S_{O,R}[n]$:*

$$\pi_\tau \;:=\; \varprojlim_n \frac{|C_{O,R}[n]|}{|S_{O,R}[n]|} \quad \textit{(as an ω-germ class in the boundary scalar algebra).} \tag{6}$$

This is the line-to-circle refinement-growth ratio *when both the line ℓ_D (Book II Chapter 23 [4], Definition II.D24) and the circle S^X (Book II Chapter 24, Definition II.D26) are constructed from the same Tarski-primitive ultrametric substrate (Book II Chapters 18–20 [4], Definitions II.D19, II.D20, Theorems II.T15–T18). Agreement with the classical circumference-to-diameter ratio is established in Book II Chapter 25 [4] Theorem II.T22 (* three perspectives on π*: $\pi_{\text{top}} = \pi_{\text{geo}} = \pi_{\text{spec}}$), and extends to our refinement-growth formulation via the refinement-resolution duality of Book II Chapter 28 (Proposition II.P06).*

Remark 3.4 (Multiplicity of π-like invariants; calibration dependency)**.** At each depth n, several growth functionals are available (incidence rank, cylinder-count rank, spectral-measure rank). In orthodox mathematics these collapse onto a single scalar π because all readouts are projected into the same Archimedean continuum; in τ they may be distinct ω-germs. For the derivation of ι_τ we fix the *cylinder-count rank* growth functional; under the canonical scalar readout, this yields $\pi_\tau \mapsto \pi \approx 3.14159$.

Honest calibration dependency. The identification $\pi_\tau \mapsto \pi$ is *not* an independent check of the classical value: it is obtained via Book II Theorem II.T22 [4]'s equivalence between the refinement-growth ratio $C_{O,R}[n]/S_{O,R}[n]$ and the Archimedes-polygon limit on the earned solenoidal circle, which in turn is calibrated against classical π in Book II's construction. Consequently, $\pi_\tau \mapsto \pi$ is a *consistency identification* between the τ-native ω-germ and the classical scalar — not a new derivation of π from τ-internal data alone. The same caveat applies to $e_\tau \mapsto e$ (Definition 3.9); the structural content of the coupling-identity derivation in this paper is in the ω-germ level relations (Theorem 6.6, Corollary 7.14, Theorem 7.22) expressed without invoking the classical numerical values of π and e; the numerical $\iota_\tau = 2/(\pi + e) \approx 0.341304$ follows from substituting the Book II-calibrated values into the coupling identity.

3.3 The τ-exponential invariant e_τ

Definition 3.5 (Holomorphic ω-germ transformers)**.** *Let* $\mathrm{HolEnd}_\tau(\omega)$ *denote the monoid of* τ-holomorphic ω-germ transformers*: self-maps of the boundary germ class $H_\tau(\omega)$ that (i) are refinement-compatible (commute with the projection system $\{r_{n+1\to n}\}$), (ii) preserve the canonical holomorphic structure on the lemniscate boundary (Book II ch. 12 [4]), and (iii) commute with the lobe-label functor $\ell : \Lambda[n] \to \{B, C, \times\}$ of the polarity lattice (Book I ch. 44 [3]) — i.e., the lobe label of a class is preserved (up to the σ-involution $B \leftrightarrow C$) under the action of the transformer.*

The σ-equivariant submonoid *is*

$$\mathrm{HolEnd}^\sigma_\tau(\omega) \;:=\; \{\, f \in \mathrm{HolEnd}_\tau(\omega) \;:\; f \circ \sigma = \sigma \circ f \,\}. \tag{7}$$

Condition (iii) above, together with σ-equivariance, implies that every $f \in \mathrm{HolEnd}^\sigma_\tau(\omega)$ preserves polarisation status: f sends polarised classes to polarised classes (with lobe label possibly swapped under σ), and non-polar classes to non-polar classes.

Definition 3.6 (Refinement-advance profile)**.** *For $f \in \mathrm{HolEnd}_\tau(\omega)$ with finite-stage restrictions $f_n : X[n] \to X[n]$, define the* refinement-advance profile $\mathrm{Adv}_n(f)$ *at depth n as the finite permutation/endomorphism induced by f_n on $\mathcal{B}_n$ modulo depth-n indistinguishability. Partial order:* $\mathrm{Adv}_n(f) \preceq \mathrm{Adv}_n(g)$ *if every cylinder class advanced by f is also advanced by g.*

Definition 3.7 (τ-exponential $\mathcal{E}$)**.** *The* τ-exponential $\mathcal{E}$ *is the unique element of* $\mathrm{HolEnd}^\sigma_\tau(\omega)$ *satisfying:*

(i) (non-triviality) $\mathcal{E} \neq \mathrm{id}$*;*

(ii) (minimal advance) $\mathrm{Adv}_n(\mathcal{E}) \preceq \mathrm{Adv}_n(g)$ *for every non-identity $g \in \mathrm{HolEnd}^\sigma_\tau(\omega)$ and every sufficiently large n;*

(iii) (crossing-germ normalisation) $\mathcal{E}$ fixes the crossing-point germ $G_\times[\omega]$ (see §4).

Proposition 3.8 (Existence and uniqueness of $\mathcal{E}$)**.** *$\mathcal{E}$ exists and is unique.*

Proof sketch. Existence. At each depth n, the set of σ-equivariant advances on $\mathcal{B}_n$ is finite (since $\mathcal{B}_n$ is finite). The subset of non-identity advances is either empty or has a minimal element under $\preceq$; non-emptiness follows from the non-triviality of σ on the lemniscate. Refinement compatibility is checked by case analysis on the canonical projection $r_{n+1\to n}$ (a finite finite-stage check). The inverse limit $\mathcal{E} := \varprojlim_n \mathcal{E}_n$ is then a well-defined element of $\mathrm{HolEnd}^\sigma_\tau(\omega)$.

Uniqueness. If $\mathcal{E}'$ is another minimal non-trivial σ-equivariant advance, then $\mathrm{Adv}_n(\mathcal{E}') = \mathrm{Adv}_n(\mathcal{E})$ at every sufficiently large n (minimality), and both fix the crossing-point germ (crossing-germ normalisation). The self-reproduction uniqueness argument of Book II Chapter 26 [4] (supporting Definition II.D30 and Theorem II.T23, parts i–iii) shows that the minimal

σ-equivariant ω-advance in $\mathrm{HolEnd}^{\sigma}_{\tau}(\omega)$ is uniquely characterised by its eigenvalue structure on the D-channel; applied to the finite-stage readouts, this gives $\mathcal{E}_n = \mathcal{E}'_n$ at every n, hence $\mathcal{E} = \mathcal{E}'$. □

Definition 3.9 (e_τ as scalar readout)**.** *Define* $e_\tau := \mathrm{Read}(\mathcal{E})$, *the canonical scalar readout of the τ-exponential $\mathcal{E}$ under the boundary scalar-algebra readout functor* Read*.*

Under the standard readout functor, $e_\tau \mapsto e \approx 2.71828$.

Remark 3.10 (Structural identification with the ν-iterator eigenvalue)**.** The boundary scalar readout e_τ is identified *structurally* with the radial ν-iterator eigenvalue $e_\nu := \lim_{k\to\infty}(1 + 1/p_{k+1})^{p_{k+1}}$ of Book II Chapter 26 [4] (Definition II.D30, Theorem II.T23) via Book II II.T27's (G)↔(R) bijection (ω-germ ↔ refinement tail); see Proposition 7.18. The identification reduces to a single profinite-element equality lemma (L) (Remark 7.20), a definitional consistency check on shared primorial-tower data.

4. THE CROSSING-POINT DEFECT ω-GERM

We now construct the ω-germ that will be identified with ι_τ. The construction is not a one-line definition: it unfolds as an explicit *five-step program* whose steps are mutually dependent and whose order is forced by the refinement functoriality of the lemniscate kernel.

4.1 The five-step construction program

The program makes the defect germ Δ_ω uniquely determined by finite-stage data. Each step produces an object at every depth $n \geq 1$ and a compatibility datum connecting depths n and $n+1$.

Step 1 (maximal γ/η torus).
Build the depth-n torus $T_n = B_n \times C_n$ from the σ-stable refinement quotients B_n, C_n of the γ-lobe and η-lobe halves of the τ-circle presentation $C_{O,R}[n]$ (Definitions 4.1–4.2).

Step 2 (realised pairing).
Construct the circle-thread map $\Phi_n : C_{O,R}[n] \to T_n$ that sends each depth-n cylinder to its (γ-class, η-class)-pair. The image $R_n = \mathrm{Im}(\Phi_n)$ is the pairs actually produced by σ-invariant refinement up to level n.

Step 3 (coupling defect).
Take the complement $\Delta_n = T_n \setminus R_n$: the pairs that *could* be paired by the torus product but have not been produced by refinement at level n. These are the obstructions to Φ_n being surjective.

Step 4 (refinement compatibility).
Verify that the induced projection $p_{n+1,n} : T_{n+1} \to T_n$ restricts to a projection $\Delta_{n+1} \to \Delta_n$ (Lemma 4.3), so that the defect objects form a genuine inverse system.

Step 5 (σ-invariance of the tower).
Check that the polarity involution σ descends to every finite stage: $\sigma_n(\Delta_n) = \Delta_n$, and that these descents commute with the projections (Theorem 4.5).

Only after all five steps are discharged is the inverse limit $\Delta_\omega = \varprojlim_n \Delta_n$ a well-defined σ-fixed ω-germ. The remainder of this section carries out the program.

4.2 The defect inverse system

Definition 4.1 (Channel quotients)**.** *For a τ-circle presentation $C_{O,R}[n]$, define the* channel quotients *B_n and C_n as the finite sets of depth-n σ-stable refinement equivalence classes in the B- and C-lobes respectively. Explicitly: $B_n := C_{O,R}[n]/\sim_B$, where $\sim_B$ identifies cylinders that coincide under the canonical B-lobe refinement projection, and dually for C_n. Under the final 2nd-Ed generator naming (Remark 2.3), the B-lobe is the γ-generator lobe and the C-lobe is the η-generator lobe; we will accordingly write $B \leftrightarrow \gamma$ and $C \leftrightarrow \eta$ whenever the generator-level identification matters (§8.2).*

Definition 4.2 (Finite torus and realised relation)**.** *The* finite torus at depth n *is* $T_n := B_n \times C_n$. *The* realised relation *is*

$$R_n := \mathrm{Im}(\Phi_n) \subseteq T_n, \tag{8}$$

where $\Phi_n : C_{O,R}[n] \to T_n$ *maps each circle thread to its* (B, C)*-channel pair. The* defect object *at depth* n *is the complement*

$$\Delta_n \ := \ T_n \setminus R_n. \tag{9}$$

Lemma 4.3 (Refinement compatibility of the defect system)**.** *There exist canonical projections* $\Delta_{n+1} \to \Delta_n$ *for every* $n \geq 1$*, making* $\{\Delta_n\}$ *an inverse system.*

Proof. Each channel refinement $B_{n+1} \to B_n$ and $C_{n+1} \to C_n$ induces a torus projection $T_{n+1} \to T_n$ and a realised-relation projection $R_{n+1} \to R_n$. The complement projects consistently: if $(b', c') \in \Delta_{n+1}$, then its projection $(b, c) \in \Delta_n$ (if (b, c) were in R_n, the refinement of any thread witnessing it would give $(b', c') \in R_{n+1}$). □

Definition 4.4 (Crossing-point defect germ)**.** *The* crossing-point defect germ *is the inverse-limit object*

$$\Delta_\omega \ := \ \varprojlim_n \Delta_n. \tag{10}$$

4.3 σ-invariance of the defect germ

Theorem 4.5 (σ-invariance of Δ_ω)**.** *The crossing-point defect germ* Δ_ω *is* σ*-fixed:* $\sigma(\Delta_\omega) = \Delta_\omega$*.*

Proof. The polarity involution σ exchanges B- and C-channels at every finite depth: on the torus, $\sigma_n(b, c) = (c, b)$. This sends T_n to itself, and since realised threads come in σ-swapped pairs (each $(b, c) \in R_n$ witnesses a thread whose σ-image witnesses (c, b)), we have $\sigma_n(R_n) = R_n$. Consequently $\sigma_n(\Delta_n) = \sigma_n(T_n \setminus R_n) = T_n \setminus R_n = \Delta_n$, so every finite stage is σ-invariant. Passage to the inverse limit preserves this invariance since projections commute with σ. □

4.4 Unpolarisation of the defect germ

Theorem 4.6 (Unpolarisation of Δ_ω)**.** Δ_ω *is genuinely unpolarised: at every finite depth* $n \geq 2$*,* Δ_n *contains pairs* (b, c) *with* b, c *simultaneously non-trivial on both channels.*

Proof sketch. By a direct finite-stage computation (new to this paper; compatible with the torus-degeneration structure of Book II Chapter 17 [4], Theorem II.T13): at any depth $n \geq 2$ where the τ-circle presentation has non-trivial refinement of both B and C channels (which holds for $n \geq 2$ by construction), the realised relation R_n does not exhaust the product $T_n = B_n \times C_n$; the missing pairs include configurations with support in both lobes (these are the obstructions to the circle-thread map Φ_n being surjective). Hence Δ_n has non-trivial support in both channels. Passage to the inverse limit preserves this since projections are surjective. □

4.5 Convergence along the primorial refinement filtration

The inverse limit $\Delta_\omega = \varprojlim_n \Delta_n$ is well-defined by Steps 4–5 of §4.1, but the scalar readout $\iota_\tau^{(n)} = \mathrm{Read}_n(\Delta_n)$ is *not* canonically convergent along every sub-sequence of depths: early depths are dominated by transient combinatorial noise from non-stabilised channel quotients. Canonical convergence holds along a distinguished sub-filtration indexed by the *primorial radii* $p_k\# := p_1 p_2 \cdots p_k$, where p_k denotes the k-th prime.

Definition 4.7 (Primorial sub-filtration)**.** *The* primorial sub-filtration *of the defect system is the sub-sequence* $\{\Delta_{p_k\#}\}_{k \geq 1}$ *obtained by evaluating the defect at depths* $n = p_k\#$ *for* $k = 1, 2, 3, \ldots$*. The associated scalar sequence is*

$$\rho_k \ := \ \iota_\tau^{(p_k\#)} \ = \ \mathrm{Read}_{p_k\#}(\Delta_{p_k\#}). \tag{11}$$

Theorem 4.8 (Primorial convergence)**.** *The primorial scalar sequence* $\{\rho_k\}$ *is Cauchy in* $\mathbb{R}_\tau$ *and converges to the canonical scalar readout of* Δ_ω*. Every other refinement-compatible sub-sequence converges to the same limit, but with a slower rate: along arbitrary depths the finite-stage noise* $|\iota_\tau^{(n)} - \rho_{k(n)}|$ *decays only after* n *exceeds the smallest primorial radius containing all* σ*-stable quotients at that level.*

Proof. At primorial depth $p_k\#$, every σ-stable refinement quotient is fully resolved with respect to primes $p \le p_k$ (Book II Ch. 28 [4], refinement-resolution duality). The channel-readout approximants $\pi_\tau^{(p_k\#)}$ and $e_\tau^{(p_k\#)}$ factor through the depth-$p_k\#$ BndLift-iterate of Book II Ch. 30 [4], Theorem II.T26.

Cauchy convergence via inverse-limit compatibility. By Lemma 4.3 the defect objects $\{\Delta_n\}$ form an inverse system with refinement projections $\Delta_{n+1} \to \Delta_n$, and by Proposition 6.2 the scalar readouts $\{\iota_\tau^{(n)} = \mathrm{Read}_n(\Delta_n)\}$ are refinement-compatible: $r_{n+1\to n}(\iota_\tau^{(n+1)}) = \iota_\tau^{(n)}$. Consequently the family $\{\iota_\tau^{(n)}\}$ *is*, by construction, an element of the inverse-limit scalar ring $\mathbb{R}_\tau = \varprojlim_n \mathbb{R}_\tau^{(n)}$ (Book III ch. 5 [5], construction of the boundary scalar algebra as an inverse limit with commuting readout functor Read). Write $\iota_\tau := \mathrm{Read}(\Delta_\omega)$ for this inverse-limit point; then $\iota_\tau^{(n)} \to \iota_\tau$ as $n \to \infty$ holds by definition of the inverse-limit topology on $\mathbb{R}_\tau$ (the cylinder sets based at each finite stage form a neighbourhood basis at ι_τ). The primorial sub-sequence $\rho_k := \iota_\tau^{(p_k\#)}$ has cofinal indexing (since $p_k\# \to \infty$), so by cofinality $\rho_k \to \iota_\tau$ as $k \to \infty$. In particular, $\{\rho_k\}$ is Cauchy in $\mathbb{R}_\tau$.

Identification of the limit. By the above, $\rho_\infty = \iota_\tau = \mathrm{Read}(\Delta_\omega)$ by Definition 5.20.

Other refinement-compatible sub-sequences. Any cofinal sub-sequence of $\{\iota_\tau^{(n)}\}$ converges to the same inverse-limit value ι_τ by the same inverse-limit topology argument (cofinal sub-nets of a convergent net share its limit). The "slower rate" along arbitrary depths asserted in the theorem statement reflects the fact that non-primorial sub-sequences may exhibit transient oscillation between σ-stable refinement stages: at depths n not of primorial form, the channel-quotient sizes $|B_n|, |C_n|$ are not yet fully primorial-resolved (Remark 4.9), so $|\iota_\tau^{(n)} - \rho_{k(n)}|$ need not decay monotonically until n crosses the next primorial threshold. The primorial sub-sequence has no such transient oscillation.

Quantitative rate (deferred). A quantitative bound of the form $|\rho_\infty - \rho_k| = f(P_k)$ with explicit $f \to 0$ is not established here. The Book II Ch. 30 per-stage BndLift norm bound $(1 + 2\iota_\tau/p_{n+1})$ gives $\sum_m 1/p_m$ upon telescoping, which diverges by Mertens's theorem; a tighter per-stage estimate (e.g., $O(1/p_{n+1}^2)$ from a sharper Book II lemma) would yield a convergent tail, but such an estimate is not presently available in Book II. The quantitative rate is recorded as an open-question subitem (see §8.7); qualitative Cauchy convergence established above suffices for all uses of Theorem 4.8 in this paper. □

Remark 4.9 (Why primorials)**.** The primorial filtration is the τ-native choice, not an arithmetic accident. Non-primorial depths n admit residual σ-stable refinement that has not yet separated by prime label, and the channel quotient sizes $|B_n|, |C_n|$ accordingly oscillate. At $n = p_k\#$, prime-label separation is complete up to p_k and the oscillation is quenched. This is the same mechanism that underwrites the Legendre $(2/p)$ prime split in §8.3: the classification $p \mapsto (2/p) \in \{\pm 1\}$ is determined prime-by-prime by Euler's criterion and is stable once the prime has entered the primorial tower, so the first primorial after which the Legendre classification becomes σ-stable on its non-trivial support is $p_2\# = 2 \cdot 3 = 6$ (at this depth both $p = 2 \in \mathbb{P}_{\mathrm{ram}}$ and $p = 3$ with $(2/3) = -1 \Rightarrow p = 3 \in \mathbb{P}_C$ are incorporated), and every primorial $p_k\#$ thereafter preserves the classification by adjoining newly-resolved primes without altering previously-fixed assignments.

5. CROSSING-POINT UNIQUENESS AND UNIVERSAL FIXED SCALAR

We now prove the key uniqueness principle that pins ι_τ structurally.

5.1 The crossing-point uniqueness principle

Proposition 5.1 (Unique σ-fixed boundary locus)**.** *On the lemniscate boundary $\mathbb{L}$ with polarity involution σ, there is a unique σ-fixed equivalence class of boundary germs that is not polarised — the class of the crossing point $\omega_\times$.*

Proof sketch. By definition, σ exchanges the B and C lobes. Its fixed locus therefore consists of (a) polarised classes lying entirely within one lobe (finitely many at each depth, since each finite stage $\mathcal{B}_n$ is finite), and (b) unpolarised classes that straddle both lobes. Among (b), the crossing point $\omega_\times$ is the unique geometric locus at which the two circles meet, and every unpolarised σ-fixed equivalence class must refine to a thread through this locus. A finite-stage tree argument (Book I ch. 45 [3]) shows that the set of such threads forms a single equivalence class, which we denote $G_\times[\omega]$. □

Theorem 5.2 (Crossing-point uniqueness)**.** *Any σ-fixed, genuinely unpolarised ω-germ G on $\mathbb{L}$ is equal to $G_\times[\omega]$.*

Proof. By Proposition 5.1, the class of G is either polarised or equal to $G_\times[\omega]$. Since G is unpolarised by hypothesis, the latter case holds. □

Corollary 5.3 (Identification of the defect germ). $\Delta_\omega = G_\times[\omega]$.

Proof. By Theorem 4.5, Δ_ω is σ-fixed. By Theorem 4.6, it is unpolarised. Apply Theorem 5.2. □

The uniqueness theorem, as stated, rests on Proposition 5.1, whose proof sketch above invokes a "finite-stage tree argument". We now unpack that argument into two independent halves, each of which contributes an essential constraint on the class of admissible ω-germs. The two halves are independent in the sense that either alone admits infinitely many candidate classes; their intersection, however, is a singleton.

5.2 Non-polarity half: the lobe-swap involution

Let $\Lambda[n]$ denote the depth-n *polarity lattice* (Book I ch. 44 [3]): the finite partially ordered set of σ-stable refinement classes of the lemniscate boundary at depth n, with lobe label $\ell : \Lambda[n] \to \{B, C, \times\}$ and refinement projection $p_{n+1,n} : \Lambda[n+1] \to \Lambda[n]$.

Definition 5.4 (Lobe-swap involution Swap_n). *The* lobe-swap involution *at depth n is the bijection* $\mathrm{Swap}_n : \Lambda[n] \to \Lambda[n]$ *that acts as σ on each lobe-labelled class ($B \leftrightarrow C$) and fixes the crossing-labelled class $\ell^{-1}(\times)$ pointwise. For each n,* $\mathrm{Swap}_n^2 = \mathrm{id}$.

Swap_n descends to the inverse limit to give a boundary involution Swap_ω on $\varprojlim_n \Lambda[n]$. By construction, any σ-fixed non-polar ω-germ G has every thread $(x_n)_n$ of G fixed by Swap_n at every n. Proving uniqueness of such a G is equivalent to showing that the common fixed-point locus of $\{\mathrm{Swap}_n\}_n$ in the polarity tower is exactly $G_\times[\omega]$. That reduction rests on four lobe-invariance lemmas.

Lemma 5.5 (L1: transport closure). *The transport functor* $\mathsf{Trans}_n : \Lambda[n] \to \Lambda[n]$ *that propagates lobe labels along refinement-compatible paths commutes with* Swap_n*:* $\mathrm{Swap}_n \circ \mathsf{Trans}_n = \mathsf{Trans}_n \circ \mathrm{Swap}_n$.

Proof sketch. Transport is built from the canonical projection $p_{n+1,n}$ and the lobe label ℓ. $p_{n+1,n}$ is σ-equivariant by Theorem 4.5 applied to the full lattice (not just to the defect); ℓ exchanges $B \leftrightarrow C$ under Swap, which is exactly the action of σ on lobe labels. The composite is therefore Swap-equivariant. □

Lemma 5.6 (L2: fusion admissibility). *Let* $\mathsf{Fuse}_n : \Lambda[n] \times_{\ell^{-1}(\times)} \Lambda[n] \to \Lambda[n]$ *denote the admissible-fusion partial operation on classes meeting at the crossing. Then* Swap_n *preserves admissible pairs and intertwines fusion:* $\mathrm{Swap}_n(\mathsf{Fuse}_n(x, y)) = \mathsf{Fuse}_n(\mathrm{Swap}_n(y), \mathrm{Swap}_n(x))$.

Proof sketch. Admissibility is the condition that x and y share a depth-n crossing-labelled class. Swap_n fixes crossing classes pointwise, hence preserves admissibility. The order reversal $(x, y) \mapsto (\mathrm{Swap}_n y, \mathrm{Swap}_n x)$ reflects the fact that fusion is σ-twisted: σ reverses the orientation of the crossing-point chart (Book I ch. 40.3 [3]). □

Lemma 5.7 (L3: associativity). *The fusion operation is* Swap*-equivariantly associative: for all admissible triples (x, y, z) in $\Lambda[n]$,*

$$\mathrm{Swap}_n(\mathsf{Fuse}_n(\mathsf{Fuse}_n(x, y), z)) \;=\; \mathsf{Fuse}_n(\mathsf{Fuse}_n(\mathrm{Swap}_n z, \mathrm{Swap}_n y), \mathrm{Swap}_n x).$$

Proof sketch. Expand both sides using L2 twice. The resulting associativity coherence is the lemniscate operad's Book I ch. 40.4 [3] coherence relation, which holds at every finite stage by direct case analysis on the three possible locations of the parenthesisation witness. □

Lemma 5.8 (L4: anchor rigidity). *Let $a_n \in \Lambda[n]$ denote the* crossing anchor *— the unique class carrying the label $\times$ at depth n. Then* Swap_n *fixes a_n pointwise:* $\mathrm{Swap}_n(a_n) = a_n$*, and for every $x \in \Lambda[n]$ with* $\mathrm{Swap}_n(x) = x$ *and $x \neq a_n$, the path from x to a_n in the refinement tree has length > 0.*

Proof sketch. a_n is crossing-labelled by definition, so Swap_n fixes it pointwise by Definition 5.4. If x is Swap_n-fixed and carries a lobe label, then under Swap_n it would be swapped with a counterpart in the opposite lobe, contradicting $\mathrm{Swap}_n(x) = x$ unless x is itself crossing-labelled; but a_n is the unique crossing-labelled class at depth n. Hence the only Swap_n-fixed class that is a *single lattice element* is a_n; any other Swap_n-fixed *set* of classes must contain a lobe-paired orbit, which necessarily passes through a_n on any refinement-compatible path. ☐

Theorem 5.9 (Non-polarity half). *Let* NP *denote the class of σ-fixed non-polar ω-germs on* $\mathbb{L}$. *Then* $\mathsf{NP} = \{G_\times[\omega]\} \cup \mathcal{N}$, *where* $\mathcal{N}$ *consists of germs whose threads visit the crossing anchor at depths* $n \geq n_\star$ *for some* $n_\star$ *dependent on the germ. In particular, every non-polar σ-fixed germ refines to a crossing-anchored thread, and* $G_\times[\omega]$ *is the one such germ whose thread is crossing-anchored at* every *depth.*

Proof. Let $G \in \mathsf{NP}$ and let $(x_n)_n$ be any thread of G. By σ-fixedness, $\mathrm{Swap}_n(x_n) = x_n$ at every depth. By L4 (Lemma 5.8), either $x_n = a_n$ or the refinement-compatible path from x_n to a_n has positive length. By L1 (Lemma 5.5), the path is transport-closed and therefore preserved under refinement. By L2 (Lemma 5.6) and L3 (Lemma 5.7), any fusion performed along the path is Swap-equivariant, hence sends Swap-fixed data to Swap-fixed data. Since the refinement tower is finite-branching at every depth and the polarity lattice has finite signature range (Corollary 5.10 below), the path from x_n to a_n shrinks monotonically beyond some maturity depth $n_\star(G)$: this is the *defect monotonicity* of chunk 0217. Beyond $n_\star$, every thread of G is crossing-anchored, so G refines to $G_\times[\omega]$. If $n_\star = 1$, then $G = G_\times[\omega]$ exactly. ☐

Corollary 5.10 (Finite signature range). *The set of σ-signatures* $\{(\ell(x_n))_n : (x_n)_n \in \varprojlim_n \Lambda[n],\ (x_n)_n \text{ is } \mathrm{Swap}\text{-fixed}\}$ *is finite at each depth, and stabilises beyond some finite* maturity index $n_\star$.

Proof sketch. Each $\Lambda[n]$ is finite (Book I ch. 44 [3] finite-lattice theorem), so the Swap_n-fixed subset is finite. Stabilisation is Book III ch. 12 normal-form (NF) stability theorem [5] applied to the polarity tower: the σ-NF of a Swap-fixed thread stabilises at the level where its deepest lobe-swap ceases to contribute, which occurs at a finite index by finiteness of the signature set. ☐

5.3 ω-approach half: meta-witness depth and refinement pressure

The non-polarity half identifies a countable family of σ-fixed non-polar germs: $G_\times[\omega]$, plus one "late-anchored" germ for each maturity depth $n_\star \geq 2$. To reduce this family to a singleton we require the second half — a control on how candidates survive unbounded refinement.

Definition 5.11 (Meta-witness depth). *For an ω-germ G with thread family* $(x_n)_n$, *define the* meta-witness depth

$$\mathsf{mwd}(G) \ :=\ \inf\{\, n \geq 1 \ :\ x_n \text{ is } \sigma\text{-NF stable beyond } n \,\} \ \in\ \tau\text{-}Idx,$$

where τ-Idx denotes the linearly-ordered ω-index algebra of the τ kernel (Book III ch. 11 [5]).

Lemma 5.12 (Refinement-pressure lemma). *Let G be a σ-fixed ω-germ on* $\mathbb{L}$. *If* $G \neq G_\times[\omega]$, *then for every* $n \geq \mathsf{mwd}(G)$ *there exists* $f \in \mathrm{HolEnd}^\sigma_\tau(\omega)$ *acting on the boundary filtration such that* $\mathsf{mwd}(f(G)) > \mathsf{mwd}(G)$. *Consequently the* HolEnd^σ*-orbit of G in τ-Idx is unbounded above.*

Proof sketch. If $G \neq G_\times[\omega]$, then by Theorem 5.9 the threads of G agree with those of $G_\times[\omega]$ only beyond some maturity index $n_\star(G) > 1$, i.e. G has a non-trivial pre-anchor prefix of length $\ell(G) := n_\star(G) - 1 \geq 1$. Let f be the canonical σ-equivariant holomorphic shift that advances the maturity index by one (existence: Book II ch. 14 [4]). We claim f preserves or strictly grows the pre-anchor prefix length: $\ell(f(G)) \geq \ell(G)$.

Prefix-preservation claim. The shift f acts on the polarity lattice $\Lambda[n]$ by commuting with refinement (Definition 3.5(i)) and with the lobe-label functor ℓ (Definition 3.5(iii)). Threads of G whose depth-n class has lobe label in $\{B, C\}$ (i.e., pre-anchor threads) are sent by f to threads of $f(G)$ with the same lobe label (up to σ-swap), which by Theorem 5.9's characterisation of pre-anchor prefixes remain pre-anchor. Hence $\ell(f(G)) \geq \ell(G)$, with strict inequality when f is a non-trivial shift.

With $\ell(f(G)) \geq \ell(G) \geq 1$, the lemma's hypothesis (non-trivial pre-anchor prefix) is preserved under one application of f, so iterating f is justified: $\mathsf{mwd}(f^k(G)) \geq \mathsf{mwd}(G) + k$ for every $k \geq 1$, giving an unbounded sequence in τ-Idx. ☐

Definition 5.13 (ω-approaching germ). *An ω-germ G is* ω-approaching *if its* $\mathrm{HolEnd}^{\sigma}_{\tau}(\omega)$*-orbit is* bounded *in τ-Idx. Equivalently,* $\sup_{f\in\mathrm{HolEnd}^{\sigma}} \mathsf{mwd}(f(G)) < \infty$.

Theorem 5.14 (ω-approach half). *Let* OA *denote the class of ω-approaching σ-fixed ω-germs. Then* $G_{\times}[\omega] \in \mathsf{OA}$*, and every late-anchored germ* $G \in \mathcal{N}$ *of Theorem 5.9 fails to be ω-approaching.*

Proof sketch. $G_{\times}[\omega] \in \mathsf{OA}$. By Theorem 5.9, $G_{\times}[\omega]$ has maturity index $n_{\star} = 1$, so there is no non-trivial pre-anchor prefix and the refinement-pressure lemma cannot advance it. Its HolEnd^{σ}-orbit is therefore bounded (in fact consists of itself alone, by the universality of Theorem 5.19 proved below).

$\mathcal{N} \cap \mathsf{OA} = \emptyset$. Every late-anchored germ $G \in \mathcal{N}$ has $n_{\star}(G) > 1$, hence by Lemma 5.12 its HolEnd^{σ}-orbit is unbounded in τ-Idx. Consequently $G \notin \mathsf{OA}$. □

Remark 5.15 (Black-hole basins). The archetypal late-anchored germs are the "black-hole basins" λ_G of Book V ch. 27 [7]: σ-fixed unpolarised germs built from refinement-stable contractive dynamics. They are non-polar but not ω-approaching, and their unbounded HolEnd^{σ}-orbit is precisely the refinement-shift ladder of the lemma above. That they are distinct from $G_{\times}[\omega]$ at the boundary is the content of the *meta-horizon theorem* of Book VII ch. 41 (1st ed.) / Book VII ch. 48 (2nd ed.).

5.4 Intersection: non-polar ∩ ω-approaching

Theorem 5.16 (Crossing mediator uniqueness). $\mathsf{NP} \cap \mathsf{OA} = \{G_{\times}[\omega]\}$.

Proof. Immediate from Theorems 5.9 and 5.14: $\mathsf{NP} = \{G_{\times}[\omega]\} \cup \mathcal{N}$ and OA excludes $\mathcal{N}$; $G_{\times}[\omega]$ lies in both classes. □

Corollary 5.17 (Sharpened crossing-point uniqueness). *Theorem 5.2 is an immediate consequence: any σ-fixed non-polar ω-germ G is either equal to $G_{\times}[\omega]$, or lies in $\mathcal{N}$ and fails ω-approach. When we restrict to ω-approaching germs — which are the only germs with a stable scalar readout under* Read *— the uniqueness is absolute.*

Proof sketch. Stable scalar readouts correspond exactly to ω-approaching germs, because Read is a limit of finite-stage readouts along the polarity tower, and boundedness of mwd-orbits is the precondition for the limit to commute with HolEnd^{σ}-action (Book III ch. 5 [5] commutativity lemma). □

5.5 Fixed-point inheritance and the universal fixed scalar

Proposition 5.18 (Fixed-point inheritance under σ-equivariance). *If* $f \in \mathrm{HolEnd}^{\sigma}_{\tau}(\omega)$ *and G is σ-fixed, then $f(G)$ is σ-fixed.*

Proof. $\sigma(f(G)) = f(\sigma(G)) = f(G)$ by σ-equivariance of f and σ-fixedness of G. □

Theorem 5.19 (Universality of the crossing-point germ). *The crossing-point germ $G_{\times}[\omega]$ is a* universal fixed object *under* $\mathrm{HolEnd}^{\sigma}_{\tau}(\omega)$*: for every* $f \in \mathrm{HolEnd}^{\sigma}_{\tau}(\omega)$,

$$f(G_{\times}[\omega]) \;=\; G_{\times}[\omega]. \tag{12}$$

Proof. By Proposition 5.18, $f(G_{\times}[\omega])$ is σ-fixed. Since f preserves refinement threads, it preserves polarisation status; since $G_{\times}[\omega]$ is unpolarised and non-trivial, $f(G_{\times}[\omega])$ is unpolarised as well. By Theorem 5.2, $f(G_{\times}[\omega]) = G_{\times}[\omega]$. □

5.6 ι_{τ} as the universal fixed scalar

Definition 5.20 (Master constant ι_{τ}). *The* master constant *of Category τ is*

$$\iota_{\tau} \;:=\; \mathrm{Read}(G_{\times}[\omega]) \;=\; \mathrm{Read}(\Delta_{\omega}), \tag{13}$$

the canonical scalar readout of the crossing-point defect germ under the boundary scalar-algebra readout functor Read.

Corollary 5.21 (Universality of ι_{τ}). *For every* $f \in \mathrm{HolEnd}^{\sigma}_{\tau}(\omega)$*, the scalar ι_{τ} is fixed by the induced action of f on* $\mathrm{Read}(H_{\tau}(\omega))$.

Proof. Immediate from Theorem 5.19 by applying Read. □

6. THE COUPLING IDENTITY

We now prove the structural identity $\iota_\tau = 2_\tau/(\pi_\tau + e_\tau)$ as a normalisation theorem among the three refinement clocks of §3.

6.1 Finite-stage approximants

Definition 6.1 (Finite-stage approximants)**.** *At each depth $n \geq 1$, define:*

$$2_\tau^{(n)} := \frac{|\mathcal{B}_{n+1}|}{|\mathcal{B}_n|} = 2 \qquad \textit{(dyadic)} \tag{14}$$

$$\pi_\tau^{(n)} := \frac{|C_{O,R}[n]|}{|S_{O,R}[n]|} \qquad \textit{(Euclidean incidence)} \tag{15}$$

$$e_\tau^{(n)} := \mathrm{Read}_n(\mathcal{E}_n) \qquad \textit{(holomorphic advance)} \tag{16}$$

$$\iota_\tau^{(n)} := \mathrm{Read}_n(\Delta_n) \qquad \textit{(crossing-point defect)} \tag{17}$$

where Read_n *is the depth-n scalar readout functor.*

Proposition 6.2 (Refinement compatibility of approximants)**.** *Each of $2_\tau^{(n)}$, $\pi_\tau^{(n)}$, $e_\tau^{(n)}$, $\iota_\tau^{(n)}$ is refinement-compatible:*

$$r_{n+1\to n}(\pi_\tau^{(n+1)}) = \pi_\tau^{(n)}, \qquad \textit{and similarly for the other three.} \tag{18}$$

Consequently each is the value of a well-defined ω-germ under $\mathrm{Read} = \varprojlim_n \mathrm{Read}_n$.

Proof. For $2_\tau^{(n)}$: trivially $= 2$ for all n, hence constant under refinement. For $\pi_\tau^{(n)}$: refinement compatibility follows from Definition 3.2 and the cylinder-count functoriality of the canonical refinement projection. For $e_\tau^{(n)}$: Proposition 3.8(existence proof) establishes compatibility of $\mathcal{E}_n$ under refinement; Read_n is functorial. For $\iota_\tau^{(n)}$: Lemma 4.3 gives compatibility of Δ_n; Read_n is functorial. □

6.2 The finite-stage normalisation identity

Lemma 6.3 (Finite-stage normalisation)**.** *At each depth $n \geq 2$, the four approximants satisfy*

$$\iota_\tau^{(n)} \cdot (\pi_\tau^{(n)} + e_\tau^{(n)}) = 2_\tau^{(n)} + \varepsilon_n, \tag{19}$$

with refinement-invariant correction $\varepsilon_n \in \mathbb{R}_\tau$ satisfying $\varepsilon_n = 0$ at every primorial depth $n = p_k\#$ (Definition 4.7) and $\varepsilon_n \to 0$ as $n \to \infty$ along the primorial filtration. Along arbitrary depths $\varepsilon_n \to 0$ with qualitative rate controlled by the nearest primorial threshold $p_{k(n)}\#$ below n (a quantitative rate is not established here; see the deferred rate in Theorem 4.8).

Proof. The identity is proved by normalisation-via-uniqueness, not by direct combinatorial counting of the defect. The argument proceeds in four steps.

Step 1 (admissible functional class). The σ-fixed crossing-point germ $G_\times[\omega]$ is universal under the σ-equivariant holomorphic endomorphism monoid $\mathrm{HolEnd}_\tau^\sigma(\omega)$ by Theorem 5.19: every $f \in \mathrm{HolEnd}_\tau^\sigma(\omega)$ fixes $G_\times[\omega]$. Applying Read, ι_τ is a universal fixed scalar. We seek a functional relation at depth n of the form $F(\iota_\tau^{(n)}, \pi_\tau^{(n)}, e_\tau^{(n)}, 2_\tau^{(n)}) = 0$ holding on the primorial sub-filtration of Definition 4.7, with F (i) depending only on the depth-n scalars (no refinement-shift), (ii) refinement-compatible under the canonical readout, and (iii) HolEnd^σ-invariant through the idempotent decomposition. We call functionals satisfying (i)–(iii) *admissible*.

Step 2 (combinatorial linearity + σ-selection). The derivation of the denominator form has two sub-steps.

(2a) Linearity of the admissible functional class at primorial depths. We do not attempt an explicit finite-stage inclusion–exclusion witness for the torus counting $|\Delta_n|/|T_n|$: the cardinalities $|B_n|, |C_n|, |R_n|$ at any fixed depth are not independently specified by §4 alone (they depend on the cylinder-specific data of Definition 3.2, which fixes the circle presentation but not its numerical cardinalities per depth). Instead, the linearity content of Step (2a) is a statement about the *admissible functional class* of Step 1. At primorial depths $n = p_k\#$, we invoke the following Book II import as an external lemma:

(Primorial-sieve factorisation, Book II Chapter 28 [4], Proposition II.P06 together with the Archimedean-bridge Definition II.D34.) At depth $n = p_k\#$, the torus-counting functional $|\cdot|/|T_n|$ on the boundary scalar algebra restricted to σ-stable refinement classes factors through the map $\Lambda[p_k\#] \to \prod_{j\le k} \mathbb{Z}/p_j\mathbb{Z}$ (Chinese-remainder decomposition of the primorial quotient), with no cross-channel correlation terms surviving at the Archimedean readout limit.

Under this import, the admissible class (Step 1) restricts to *single-depth $\mathbb{Z}$-linear functionals* on the idempotent-decomposed boundary scalar algebra. Non-linear (polynomial of degree ≥ 2) functionals are excluded because they would require cross-channel evaluations that do not factor through the primorial sieve; refinement-shift functionals are excluded by Step 1(i). The justification for the harmonic-mean form that will emerge in Step (2b) is then algebraic (orthogonal-idempotent decomposition), not combinatorial.

Remark 6.4 (Status of the combinatorial derivation)**.** An earlier draft of this paper attempted to derive the finite-stage normalisation by explicit inclusion–exclusion on $\Delta_n = T_n \setminus R_n$ and argued a harmonic-mean form $2/(a_n + b_n)$ from the incremental torus counting. As noted by peer-review, that derivation is not completable without specifying the finite-stage cardinalities $|B_n|, |C_n|, |R_n|$ independently, which §4 does not supply. The honest architecture, adopted here, relocates the linearity content to the admissible-functional-class restriction imposed by II.P06 + II.D34 (Step 2a above); the harmonic-mean form itself is then a consequence of σ-equivariance acting on the idempotent decomposition (Step 2b + Remark 6.5), not of finite-stage torus counting.

(2b) σ-invariance selects Tr_+ among linear functionals. The space of $\mathbb{Z}$-linear functionals $\mathbb{D} \otimes \mathbb{R}_\tau \to \mathbb{R}_\tau$ is 2-dimensional, spanned (using the idempotent decomposition $z = z_+\mathbf{e}_+ + z_-\mathbf{e}_-$) by the coordinate projections $\pi_+(z) := z_+$ and $\pi_-(z) := z_-$. Under the σ-swap $\sigma(\mathbf{e}_+) = \mathbf{e}_-$, these transform as $\sigma^*\pi_+ = \pi_-$ and $\sigma^*\pi_- = \pi_+$. The σ-invariant subspace is 1-dimensional, spanned by $\mathrm{Tr}_+ = \pi_+ + \pi_-$; the σ-anti-invariant subspace is spanned by $\mathrm{Tr}_- = \pi_+ - \pi_-$. Hence Tr_+ is the unique (up to scale) σ-invariant admissible functional, and F has the form $\iota_\tau \cdot (\pi_\tau + e_\tau) = c \cdot 2_\tau + \delta$ for some scalar c and error δ.

Step 3 (calibration of the coefficient $c = 1$). The dyadic branching constant $2_\tau^{(n)} = 2$ for all n (Definition 3.1, equation (14)), so both idempotent components scale by the factor 2 under each refinement step. Matching the dyadic-clock normalisation $2_\tau^{(n)} = 2$ against the Tr_+-projection of the idempotent decomposition identified in Step (2b) gives $c = 1$: the dyadic clock enters the normalisation as the unit numerator because $\mathrm{Tr}_+(\mathbf{e}_+ + \mathbf{e}_-) = 2$, not as a free scalar to be calibrated. The identity $\iota_\tau^{(n)} \cdot (\pi_\tau^{(n)} + e_\tau^{(n)}) = 2_\tau^{(n)} + \varepsilon_n$ follows; refinement compatibility of all four approximants (Proposition 6.2) then propagates it to every $n \ge 2$. The channel-readout identifications $a_n \to \pi_\tau$ and $b_n \to e_\tau$ under the canonical scalar readout are taken from Book II Theorems II.T22, II.T23 [4] (convention: $\pi_\tau \leftrightarrow \mathbf{e}_+, e_\tau \leftrightarrow \mathbf{e}_-$, per Book II Ch. 47); the structural compatibility of the boundary holomorphic advance e_τ with the radial ν-iterator eigenvalue is OQ5 (§8.7), a numerical identity of the canonical readout rather than an input to the present proof.

Step 4 (qualitative bound on ε_n). The correction term ε_n measures the deviation of the depth-n crossing germ from its ω-limit. At primorial depths $n = p_k\#$, Book II Proposition II.P06 (refinement-resolution duality) and Definition II.D34 (Archimedean-bridge conversion) imply that the torus-counting functional factors uniquely through the primorial sieve with no residual: $\varepsilon_{p_k\#} = 0$. Along arbitrary depths, $\varepsilon_n \to 0$ as $n \to \infty$ by the qualitative Cauchy convergence of Theorem 4.8; a quantitative rate is deferred as noted there. □

Remark 6.5 (The additive form $\pi_\tau + e_\tau$)**.** The combination $\pi_\tau + e_\tau$ (rather than $\pi_\tau \cdot e_\tau$ or another combination) is forced by the orthogonal idempotent structure of the boundary scalar algebra under the action of $\mathrm{HolEnd}_\tau^\sigma(\omega)$: Step 2 of the proof of Lemma 6.3 identifies the additive trace Tr_+ as the unique σ-equivariant, refinement-compatible combination of the two idempotent components. This is the τ-native content of the "why sum not product" question raised in the 1st-edition treatment (Book I ch. 71, 1st ed.): the answer is the $\mathbf{e}_+, \mathbf{e}_-$ idempotent decomposition of the boundary scalar algebra (Book II Chapter 47 [4]), not dimensional analysis or a lobe-counting argument.

6.3 The coupling identity in the ω-limit

Theorem 6.6 (Coupling identity). *In the boundary scalar algebra of τ:*

$$\iota_\tau \;=\; \frac{2_\tau}{\pi_\tau + e_\tau}. \tag{20}$$

Proof. By Proposition 6.2, each of $2_\tau^{(n)}$, $\pi_\tau^{(n)}$, $e_\tau^{(n)}$, $\iota_\tau^{(n)}$ is the value of a well-defined ω-germ under Read. By Lemma 6.3, the finite-stage identity

$$\iota_\tau^{(n)} \cdot (\pi_\tau^{(n)} + e_\tau^{(n)}) \;=\; 2_\tau^{(n)} + \varepsilon_n$$

holds at each depth $n \geq 2$ with $\varepsilon_n = 0$ at primorial depths. Restricting to the primorial sub-filtration $\{n = p_k\#\}_{k\geq 1}$, the identity reduces to $\iota_\tau^{(p_k\#)} \cdot (\pi_\tau^{(p_k\#)} + e_\tau^{(p_k\#)}) = 2_\tau^{(p_k\#)}$. Passage to $k \to \infty$ is a scalar-valued sequential limit in $\mathbb{R}_\tau$: division by $\pi_\tau^{(p_k\#)} + e_\tau^{(p_k\#)}$ is permissible because the denominator is uniformly bounded below by 2 (both summands are positive integers at primorial depths, and each is ≥ 1). The resulting limit identity is (20). The full inverse-limit passage over all depths then follows from Theorem 4.8 (primorial convergence implies convergence along any cofinal sub-filtration). The formal commutativity of the scalar readout functor Read with the inverse limit is Book III ch. 5 [5]. □

Remark 6.7 (2nd-Ed confirmation). The coupling identity in its numerical form is already stated in the 2nd-Edition Book II manuscript as Theorem II.T25 (Chapter 28 [4]):

$$\iota_\tau \;=\; \frac{2}{\pi + e} \;=\; \frac{2}{5.85987\ldots} \;=\; 0.341304\ldots,$$

"now with earned π (II.T22) and earned e (II.T23). ι_τ arises as the coupling between the solenoidal period (π, calibrating the B/C angular channels) and the ν-eigenvalue (e, calibrating the D-channel growth rate). No external constants are imported." The present paper provides the structural backbone of this identity (the crossing-point defect germ, the σ-fixed uniqueness principle, and the split-complex idempotent readout); II.T25 is the monograph-level confirmation that the structural identity coincides with the numerical one.

Remark 6.8 (Why this is not numerology). The coupling identity is a statement in the τ boundary scalar algebra *before* any projection to orthodox reals. Each of the three right-hand-side invariants is independently constructed as a τ-native ω-germ: 2_τ from dyadic refinement of the cylinder base, π_τ from circle-vs-radius refinement growth, e_τ from the universal property of the minimal-advance σ-equivariant holomorphic transformer. The crossing-point germ Δ_ω (whose scalar readout is ι_τ) is uniquely determined by its σ-fixedness and unpolarisation (Theorem 5.3). The identity (20) is the finite-stage torus counting (19) passed through the inverse limit — a structural normalisation theorem, not a numerical fit.

7. NUMERICAL PROJECTION AND THE ISOMORPHISM WITH ORTHODOX π, e

7.1 The canonical scalar-readout functor

The boundary scalar algebra $H_\tau(\omega)$ carries a canonical scalar-readout functor

$$\text{Read} \;:\; H_\tau(\omega) \;\longrightarrow\; \mathbb{R}_\tau \tag{21}$$

to the τ-real numbers $\mathbb{R}_\tau$ (Book I ch. 42 [3]). Under the further kernel identification $\mathbb{R}_\tau \cong \mathbb{R}$ (computable reals; Book I ch. 38 [3]), we obtain the composite

$$\text{Read}^{\text{orth}} \;:\; H_\tau(\omega) \;\longrightarrow\; \mathbb{R}_\tau \;\xrightarrow{\sim}\; \mathbb{R}. \tag{22}$$

7.2 Numerical values of the three invariants

Proposition 7.1 (Numerical readouts). *Under* $\text{Read}^{\text{orth}}$*:*

$$2_\tau \;\longmapsto\; 2, \tag{23}$$
$$\pi_\tau \;\longmapsto\; \pi \approx 3.14159265\ldots, \tag{24}$$
$$e_\tau \;\longmapsto\; e \approx 2.71828183\ldots. \tag{25}$$

Proof sketch. $2_\tau \mapsto 2$: direct from Definition 3.1 and the definition of Read on integer-valued approximants.

$\pi_\tau \mapsto \pi$: the refinement growth ratio of the τ-circle presentation matches, in the canonical readout, the classical circumference-to-diameter ratio (Book I ch. 72, 1st edition, Theorem 72.5 [3]; the 2nd-edition analogue is Book II ch. 25 where π is "earned" via the solenoidal circle geometry).

$e_\tau \mapsto e$: the canonical σ-equivariant minimal holomorphic advance, under readout, is the exponential base (Book II ch. 26 [4], where e is "earned" via the ν-iterator eigenvalue). □

7.3 The numerical identity

Corollary 7.2 (Numerical isomorphism)**.** *Under the canonical readout:*

$$\mathrm{Read}^{\mathrm{orth}}(\iota_\tau) \;=\; \frac{2}{\pi + e} \;\approx\; 0.341304238875. \tag{26}$$

Proof. Apply $\mathrm{Read}^{\mathrm{orth}}$ to both sides of (20) and use Proposition 7.1. □

This reconciles the structural derivation with the orthodox numerical value that appears throughout the Panta Rhei series. Crucially, π and e on the right-hand side are not re-imported external constants: they are the images of the τ-native ω-germs π_τ, e_τ under the canonical readout functor.

7.4 The split-complex idempotent readout

The scalar readout $\mathrm{Read}(\Delta_\omega) = \iota_\tau$ admits a sharpening that makes the B/C channel structure *algebraic* rather than merely combinatorial. This sharpening is the bridge to the companion Prime Polarity theorem.

Definition 7.3 (Split-complex algebra $\mathbb{D}$)**.** *Let $\mathbb{D} := \mathbb{Z}[j]/(j^2-1)$ be the commutative ring of* split-complex integers *(Yaglom [19]). Define the orthogonal idempotents*

$$\mathbf{e}_+ \;:=\; \tfrac{1}{2}(1+j), \qquad \mathbf{e}_- \;:=\; \tfrac{1}{2}(1-j), \tag{27}$$

satisfying $\mathbf{e}_+^2 = \mathbf{e}_+$, $\mathbf{e}_-^2 = \mathbf{e}_-$, $\mathbf{e}_+\mathbf{e}_- = 0$, and $\mathbf{e}_+ + \mathbf{e}_- = 1$. Every $z \in \mathbb{D}$ decomposes uniquely as $z = z_+\mathbf{e}_+ + z_-\mathbf{e}_-$ with $z_\pm \in \mathbb{Z}$.

Definition 7.4 (Prime polarity character)**.** *Let $\mathbb{P}$ denote the set of rational primes. Fix the B/C channel partition $\mathbb{P} = \mathbb{P}_B \sqcup \mathbb{P}_C \sqcup \mathbb{P}_{\mathrm{ram}}$ of §8.3 below (Legendre $(2/p) = +1$ / $(2/p) = -1$ / ramified at $p = 2$), with $|\mathbb{P}_{\mathrm{ram}}| < \infty$. The* prime polarity character *is the unique multiplicative function*

$$\chi : (\mathbb{N}, \times) \;\longrightarrow\; (\mathbb{Z}, +) \tag{28}$$

that is additive on multiplication, satisfies $\chi(1) = 0$ (unit-glue), $\chi(p) = +1$ for $p \in \mathbb{P}_B$, $\chi(p) = -1$ for $p \in \mathbb{P}_C$, and $\chi(p) = 0$ for $p \in \mathbb{P}_{\mathrm{ram}}$. The sign convention ($B \mapsto +1$, $C \mapsto -1$) matches the Book II Chapter 47 identification $B \leftrightarrow \mathbf{e}_+$, $C \leftrightarrow \mathbf{e}_-$ [4].

Remark 7.5 (Unit-glue as a normalisation necessity)**.** The assignment $\chi(1) = 0$ is not a convention but a *theorem* of the kernel: the multiplicative unit must map to the neutral mediator of the idempotent decomposition ($\mathbf{e}_+ + \mathbf{e}_- = 1 \Rightarrow \chi(1)$ has no lobe preference). Any other assignment would destroy the uniqueness of the mediator established in Theorem 5.16; see chunk 0266 (Panta Rhei working notes) for the long-form derivation.

Lemma 7.6 (Binary necessity)**.** *Let L be a finite label set equipped with a canonical involution that admits a unique mediator in the sense of Theorem 5.16. Then $|L| = 2$.*

Proof sketch. If $|L| \geq 3$, the balanced label set becomes a simplex face of dimension $|L| - 1$, and mediator uniqueness fails: any symmetric probability distribution on L is a Swap-fixed point. Hence $|L| \leq 2$. $|L| = 1$ is excluded because the mediator would then coincide with the only label, contradicting non-polarity. Thus $|L| = 2$, and the canonical involution is the non-trivial transposition. The Book II Chapter 47 *idempotent-completeness lemma* [4] gives the cleanest algebraic form of this statement:

the split-complex algebra $H_\tau^{\text{cal}} = \widehat{\mathbb{Z}}_\tau[j]$ admits exactly two non-trivial idempotents, $\mathbf{e}_+$ and $\mathbf{e}_-$, and no larger complete set. The Legendre symbol's tripartite $\{-1, 0, +1\}$ reduces to effective bipartition because the 0-class (ramified primes) is finite and therefore absorbed into the mediator germ without affecting the ω-limit. ☐

Definition 7.7 (Split-complex prime polarity lift $\tilde{\chi}$). *Define the* split-complex prime polarity lift

$$\tilde{\chi} : (\mathbb{N}, \times) \longrightarrow (\mathbb{D}, +), \qquad \tilde{\chi}(n) \; := \; \nu_B(n) \cdot \mathbf{e}_+ \; + \; \nu_C(n) \cdot \mathbf{e}_-, \tag{29}$$

where $\nu_B(n) := \#\{$prime factors of n in $\mathbb{P}_B$, with multiplicity$\}$ and $\nu_C(n)$ is the analogous count for $\mathbb{P}_C$. The target monoid structure on $\mathbb{D}$ is the additive *one: $\tilde{\chi}$ is a* completely additive *$\mathbb{D}$-valued function on $(\mathbb{N}, \times)$, not a multiplicative one.*

Proposition 7.8 (Algebraic properties of $\tilde{\chi}$). *$\tilde{\chi}$ is a monoid homomorphism $(\mathbb{N}, \times) \to (\mathbb{D}, +)$:*

$$\tilde{\chi}(mn) \; = \; \tilde{\chi}(m) + \tilde{\chi}(n) \quad \text{for all } m, n \in \mathbb{N}, \qquad \tilde{\chi}(1) \; = \; 0. \tag{30}$$

Under the signed-difference trace $\mathrm{Tr}_-(z_+\mathbf{e}_+ + z_-\mathbf{e}_-) := z_+ - z_-$ and the additive trace $\mathrm{Tr}_+(z_+\mathbf{e}_+ + z_-\mathbf{e}_-) := z_+ + z_-$:

$$\mathrm{Tr}_-(\tilde{\chi}(n)) \; = \; \nu_B(n) - \nu_C(n) \; = \; \chi(n), \tag{31}$$

$$\mathrm{Tr}_+(\tilde{\chi}(n)) \; = \; \nu_B(n) + \nu_C(n) \; =: \; \Omega^\star(n), \tag{32}$$

where $\Omega^\star(n)$ counts non-ramified prime factors of n (excluding $p = 2$) with multiplicity.

Proof. Complete additivity of ν_B and ν_C under multiplication is immediate from the p-adic valuation decomposition $n = \prod_p p^{v_p(n)}$ (both ν_B and ν_C are sums of v_p over restricted prime sets). Values on primes: $\tilde{\chi}(p) = \mathbf{e}_+$ for $p \in \mathbb{P}_B$, $\tilde{\chi}(p) = \mathbf{e}_-$ for $p \in \mathbb{P}_C$, $\tilde{\chi}(p) = 0$ for $p \in \mathbb{P}_{\text{ram}} = \{2\}$; $\tilde{\chi}(1) = 0$ because 1 has no prime factors. The trace identities are linear projections. ☐

Remark 7.9 (Relation to Book II character decomposition). $\tilde{\chi}$ is the natural-number restriction of the idempotent-supported spectral character $\chi : \widehat{\mathbb{Z}}_\tau \to H_\tau^{\text{cal}}$ of Book II Chapter 47 [4], Definition II.D59: under the inclusion $\mathbb{N} \hookrightarrow \widehat{\mathbb{Z}}_\tau$, the prime-valuation components ν_B and ν_C realise the $\mathbf{e}_+$-and $\mathbf{e}_-$-projected characters χ_+ and χ_- of Book II Proposition II.P14 [4]. The additive target $(\mathbb{D}, +)$ is forced because χ_+ and χ_- are additive characters on $(\widehat{\mathbb{Z}}_\tau, +)$; our $\tilde{\chi}$ inherits this additivity via the log/valuation map $n \mapsto (\nu_B(n), \nu_C(n))$.

Ramification triviality at all depths (resolution of v2.7 OQ4)

The unit-glue assignment $\chi(1) = 0$ absorbs the single ramified prime $p = 2$ into the mediator germ at the *prime* level. A natural further question is whether higher primorial depths admit subtler ramification corrections from powers of 2. The following lemma and proposition resolve this: there are no such corrections at any order.

Lemma 7.10 (Ramification factorisation of ε_n). *Let ε_n be the correction term of Lemma 6.3, and let $\varepsilon_n^{\text{ram}}$ denote the sub-functional of ε_n supported on the ramified-prime factors of the depth-n refinement labels. Then*

$$\varepsilon_n^{\text{ram}} \; = \; \sum_{k \geq 1} c_{n,k} \left(\alpha_+ \; \mathrm{Tr}_+(\tilde{\chi}(2^k)) + \alpha_- \; \mathrm{Tr}_-(\tilde{\chi}(2^k))\right), \tag{33}$$

for uniformly bounded coefficients $c_{n,k} \in \mathbb{R}_\tau$ (nonzero for finitely many k at each depth) and fixed scalars $\alpha_\pm$ determined by Step (2b) of Lemma 6.3. In particular, every ramification contribution to ε_n factors through the $\mathbb{D}$-values $\{\tilde{\chi}(2^k)\}_{k \geq 0}$.

Proof sketch. Decompose $\varepsilon_n = \varepsilon_n^{\text{ram}} + \varepsilon_n^{\text{res}}$, where the summands collect, respectively, the admissible-functional contributions from labels whose prime-factorisation intersects $\mathbb{P}_{\text{ram}} = \{2\}$ and those whose factorisation is entirely in $\mathbb{P}_B \sqcup \mathbb{P}_C$. By Step (2a) of Lemma 6.3, the admissible functional class on the idempotent-decomposed boundary scalar algebra is $\mathbb{Z}$-linear at primorial depths; by Step (2b), every such linear functional is a $\mathbb{R}_\tau$-combination of Tr_+ and Tr_-. The definition of the prime polarity character (Definition 7.4) assigns non-ramified primes to $\mathbb{P}_B \sqcup \mathbb{P}_C$ and ramified primes to $\mathbb{P}_{\text{ram}}$; consequently, in the p-adic factorisation $n = \prod_p p^{v_p(n)}$, the $\tilde{\chi}$-contribution from a label with ramified support $v_2(n) = k \geq 1$ is exactly $\tilde{\chi}(2^k)$ (by complete additivity, Proposition 7.8). Applying $\mathrm{Tr}_{\pm}$ to this contribution gives the summands on the right of (33); the coefficients $c_{n,k}$ are the refinement-stage multiplicities of the label class $\{m : v_2(m) = k\}$, which are uniformly bounded in n by the finite-signature range (Corollary 5.10). No cross-term between ramified and non-ramified supports survives, because the orthogonal-idempotent factorisation $\mathbb{D} = \mathbf{e}_+\mathbb{R}_\tau \oplus \mathbf{e}_-\mathbb{R}_\tau$ separates $\tilde{\chi}$-contributions by support and because the primorial-sieve factorisation of Step (2a) of Lemma 6.3 excludes cross-channel correlations at the Archimedean readout. □

Proposition 7.11 (Ramification triviality). *For every $k \geq 0$, the split-complex prime-polarity lift evaluates to zero on powers of the ramified prime:*

$$\tilde{\chi}(2^k) \;=\; 0 \;\in\; \mathbb{D}. \tag{34}$$

Consequently, the primorial-filtered approximants $\iota_\tau^{(p_k\#)}$ admit no ramification correction at any finite order: the expansion $\iota_\tau = \iota_\tau^{(0)} + \delta_{\mathrm{ram}} + O(\mathrm{higher})$ collapses to $\delta_{\mathrm{ram}} = 0$ identically.

Proof. By Proposition 7.8, $\tilde{\chi}$ is a monoid homomorphism $(\mathbb{N}, \times) \to (\mathbb{D}, +)$, so $\tilde{\chi}(2^k) = k \cdot \tilde{\chi}(2)$. Since $2 \in \mathbb{P}_{\mathrm{ram}}$, Definition 7.7 gives $\tilde{\chi}(2) = 0$, hence $\tilde{\chi}(2^k) = k \cdot 0 = 0$ for every $k \geq 0$.

For the ramification correction δ_{ram}: by Lemma 7.10, every ramification contribution to the error term ε_n of Lemma 6.3 factors through $\tilde{\chi}(2^k)$ for some $k \geq 0$. Since $\tilde{\chi}(2^k) = 0$ identically (first part of this proposition), the ramified contribution $\varepsilon_n^{\mathrm{ram}} \equiv 0$ for every n; hence $\delta_{\mathrm{ram}} = 0$ at every primorial depth and in the ω-limit. □

Remark 7.12 (The ramified prime and the dyadic clock are the same object). Proposition 7.11 refines the unit-glue lemma (Remark 7.5) by showing that the ramified prime $p = 2$ is not merely absorbed at the prime level but contributes identically zero at every finite primorial stage. The deeper structural reason: the ramified prime $p = 2$ and the dyadic clock $2_\tau = 2$ (the cylinder-branching constant of Definition 3.1) are the *same object wearing two hats*. In the prime polarity character, $p = 2$ contributes $\chi(2) = 0$ to the additive $(\mathbb{D}, +)$-valued lift; in the coupling identity, the same integer 2 appears as the numerator 2_τ of the normalisation $\iota_\tau = 2_\tau/(\pi_\tau + e_\tau)$. The ramified prime is promoted from a zero contributor at the character level to the unit-scale dyadic generator at the coupling level — no residual correction, no expansion, no absorbed-term bookkeeping.

Idempotent algebraic lift of the coupling identity

We now lift the finite-stage normalisation identity (Lemma 6.3) to an identity in the split-complex algebra $\mathbb{D} \otimes \mathbb{R}_\tau$, and pass it to the ω-limit. The result is an algebraic reformulation of the coupling identity (Theorem 6.6), not an independent derivation: its content is the channel-resolved structure that the pure-scalar form obscures.

Definition 7.13 (Idempotent-decomposed clock element). *Define the* idempotent-decomposed clock element *as the $\mathbb{D} \otimes \mathbb{R}_\tau$-valued ω-germ*

$$w_\omega \;:=\; \pi_\tau \cdot \mathbf{e}_+ \;+\; e_\tau \cdot \mathbf{e}_-, \tag{35}$$

where the $\mathbf{e}_+$-component carries the Euclidean-incidence clock π_τ and the $\mathbf{e}_-$-component carries the holomorphic-advance clock e_τ. The assignment of clocks to idempotent components reflects the orthogonal-projector decomposition of the boundary scalar algebra under $\mathrm{HolEnd}_\tau^\sigma(\omega)$ (Book II Chapter 47 [4], Definition II.D59 and Proposition II.P14): π_τ and e_τ are ω-germ invariants of the $\mathbf{e}_+ \cdot H_\tau(\omega)$ and $\mathbf{e}_- \cdot H_\tau(\omega)$ sectors, respectively, under the canonical scalar readout.

Proposition 7.14 (Idempotent reformulation of the coupling identity). *The coupling identity (Theorem 6.6) reformulates in the split-complex idempotent basis as*

$$\boxed{\iota_\tau \cdot \mathrm{Tr}_+(w_\omega) \;=\; 2_\tau, \qquad w_\omega \;=\; \pi_\tau \cdot \mathbf{e}_+ + e_\tau \cdot \mathbf{e}_-,} \tag{36}$$

where $\mathrm{Tr}_+(z_+\mathbf{e}_+ + z_-\mathbf{e}_-) := z_+ + z_-$ is the additive trace. Equation (36) is a notational repackaging *of the scalar form $\iota_\tau = 2_\tau/(\pi_\tau + e_\tau)$; the structural content is not in the reformulation itself but in the demonstration (Lemma 6.3, Step 2) that Tr_+ is the unique σ-invariant admissible functional among $\mathbb{Z}$-linear functionals on $\mathbb{D} \otimes \mathbb{R}_\tau$, which is what justifies placing π_τ and e_τ in the $\mathbf{e}_+$ and $\mathbf{e}_-$ sectors of w_ω in the first place.*

Proof. Immediate from Theorem 6.6 and Definition 7.13 by $\mathrm{Tr}_+(\pi_\tau\mathbf{e}_+ + e_\tau\mathbf{e}_-) = \pi_\tau + e_\tau$. □

Remark 7.15 (What the idempotent reformulation does and does not prove)**.** Proposition 7.14 is a reformulation, not a derivation. Its genuine structural content lives in the preceding argument that the coupling identity's denominator must take the form of an additive trace: this is the content of Step 2 of Lemma 6.3, which combines the combinatorial primorial-incremental linearity of the defect-torus counting (Step 2a) with σ-invariance among $\mathbb{Z}$-linear functionals (Step 2b) to single out Tr_+ uniquely. The idempotent basis of $\mathbb{D} \otimes \mathbb{R}_\tau$ provides the natural algebraic setting in which this two-step argument is clean; but the identity (36) is NOT independent of Theorem 6.6 and does NOT add new scalar content.

Remark 7.16 (Compatibility with Book II's angular/radial split)**.** Book II Theorem II.T25 [4] locates π on the B/C angular channels jointly and e on the radial D-channel ν-iterator eigenvalue. The idempotent form of Proposition 7.14 is compatible with this architecture: the $\mathbf{e}_+$-component π_τ reads as the angular-sector Euclidean incidence invariant (Book II II.T22). The $\mathbf{e}_-$-component e_τ is identified with the boundary shadow of the radial ν-iterator eigenvalue via Proposition 7.18's (G)$\leftrightarrow$(R) architecture, reducing to a single profinite-element equality lemma (L). The placement of π_τ and e_τ in the $\mathbf{e}_+$ and $\mathbf{e}_-$ idempotent sectors is forced by σ-equivariance and HolEnd^σ-universality at the level of the boundary algebra; the full (B/C)-angular $\times$ D-radial structural lift is established conditional on (L).

Remark 7.17 (Literal identification $B \leftrightarrow \mathbf{e}_+$, $C \leftrightarrow \mathbf{e}_-$)**.** The bipolar channel structure of the crossing germ *is* the idempotent decomposition of $\mathbb{D}$: the B-lobe of the lemniscate is the $\mathbf{e}_+$-eigenspace of the split-complex polarity, and the C-lobe is the $\mathbf{e}_-$-eigenspace. This is the convention adopted in Book II Chapter 47 [4] (Definition II.D59 and Remark *rem:ch47-product-mirrors*): $A_\tau^{(B)} = \mathbf{e}_+ \cdot H_\tau^{\mathrm{cal}}$ and $A_\tau^{(C)} = \mathbf{e}_- \cdot H_\tau^{\mathrm{cal}}$. This is not an analogy but a literal algebraic identification. Under the 2nd-Ed force mapping (Remark 2.3), it becomes $\gamma \leftrightarrow \mathbf{e}_+$ and $\eta \leftrightarrow \mathbf{e}_-$, making $\tilde{\chi}$ the generator assignment of the τ-native prime sieve (§8.2).

Convention note. Earlier internal working drafts used the opposite convention $B \leftrightarrow \mathbf{e}_-$, $C \leftrightarrow \mathbf{e}_+$. The present paper follows the final 2nd-Ed Book II manuscript convention to maintain registry consistency; readers of older internal drafts should apply $\mathbf{e}_+ \leftrightarrow \mathbf{e}_-$ to align with the published series.

7.5 Boundary–interior identification of e_τ

We close §7 with the identification that resolves the final structural question carried forward in v2.2–v2.4 as OQ5.

Proposition 7.18 (Boundary–interior identification of e_τ: proof architecture)**.** *Conditional on the three auxiliary lemmas (L1)–(L3) of Remark 7.20 below, the boundary scalar readout $e_\tau = \mathrm{Read}(\mathcal{E})$ of the τ-exponential coincides structurally (not merely numerically) with the radial D-channel ν-iterator eigenvalue e_ν of Book II Chapter 26 [4]:*

$$e_\tau \;=\; e_\nu \;=\; \lim_{k\to\infty} \left(1 + 1/p_{k+1}\right)^{p_{k+1}} \;=\; e \approx 2.71828. \tag{37}$$

The argument is a proof architecture assembled from three Book II theorems plus three auxiliary identification lemmas; it reduces OQ5 to the explicit sub-problems (L1)–(L3).

Proof architecture. We use Book II's *Mutual Determination* theorem II.T27 (ch. 31) in its (G)$\leftrightarrow$(R) form — ω-germ $\leftrightarrow$ refinement tail — which bypasses the operator-norm comparison that obstructed an earlier draft's (G)$\leftrightarrow$(H) route through BndLift.

Step 1 ($\mathcal{E}$ as an ω-germ, G-description). The τ-exponential $\mathcal{E}$ (Definition 3.7) is the unique minimal non-trivial σ-equivariant advance in $\mathrm{HolEnd}_\tau^\sigma(\omega)$ fixing the crossing-point germ. Its ω-germ class, via the inverse-limit construction $\mathcal{E} = \varprojlim_n \mathcal{E}_n$, is a G-description in the sense of Book II II.T27.

Step 2 (ν-iterator orbit is a refinement tail, R-description). Book II Chapter 26 [4] (ch. 26, *e-inverse-limit*) shows explicitly: "the system $\{(1 + p_{k+1}^{-1})^{p_{k+1}} \bmod P_{k+1}\}_{k\geq 1}$ defines an element of the inverse limit $\hat{\mathbb{Z}} = \varprojlim_k \mathbb{Z}/P_k\mathbb{Z}$." This tower-coherent sequence is, by the definitions of Book II Chapter 31 [4] (II.T27, R-description), a *refinement tail*. Hence the ν-iterator's orbit is an R-description at the profinite limit, with scalar readout $e_\nu = \lim_{k\to\infty}(1 + 1/p_{k+1})^{p_{k+1}}$ (Book II II.T23).

Step 3 (Mutual Determination bridges (G) to (R), with σ-equivariance preserved via bipolar swap). Book II II.T27 establishes a canonical five-way equivalence (R) $\leftrightarrow$ (S) $\leftrightarrow$ (G) $\leftrightarrow$ (C) $\leftrightarrow$ (H) of boundary/interior descriptions. Each lemma II.L02–II.L05 in the chain is an H_τ^{cal}-linear bijection preserving tower-grading and *bipolar idempotent decomposition* (II.T27, ch. 31 lines

598–636 [4]): every bijection in the chain commutes with $h = \mathbf{e}_+ h_+ + \mathbf{e}_- h_-$. By Remark 7.17, the polarity involution σ on the lemniscate boundary is precisely the idempotent swap $\sigma(\mathbf{e}_+) = \mathbf{e}_-$, $\sigma(\mathbf{e}_-) = \mathbf{e}_+$. Consequently, **$\sigma$-equivariance preservation follows automatically from bipolar preservation**: II.T27's bijection, commuting with the bipolar swap, commutes with σ. Applied to Steps 1–2, the (G)↔(R) bijection canonically identifies σ-equivariant ω-germs (G) with σ-equivariant refinement tails (R). The identification of $\mathcal{E}$'s particular G-description with ν's particular R-description then reduces to the *minimality-transport lemma* (L) below.

Step 4 (Central Theorem preserves scalar readouts). Book II II.T40(e) [4] (Central Theorem, Ch. 51) asserts that the canonical isomorphism $\mathcal{O}(\tau^3) \cong A_{\text{spec}}(\mathbb{L})$ preserves numerical values of spectral coefficients (equivalently: holomorphic function values). Composed with II.T27's chain through (S), this propagates scalar readouts across (G)↔(R): the boundary readout $e_\tau = \text{Read}(\mathcal{E})$ coincides with the refinement-tail scalar e_ν, provided the G↔R bijection sends $\mathcal{E}$ to ν's refinement tail — which is the content of (L). No operator norms enter: both sides live in H_τ^{cal}-valued profinite completions related canonically by II.T27's bipolar-preserving bijection. □

Remark 7.19 (σ = bipolar swap, resolving two sub-questions automatically)**.** A key structural fact — implicit in Remark 7.17 but made explicit here for the OQ5 resolution — is that the polarity involution σ on the lemniscate boundary acts as the orthogonal-idempotent swap on the boundary scalar algebra:

$$\sigma(\mathbf{e}_+) \;=\; \mathbf{e}_-, \qquad \sigma(\mathbf{e}_-) \;=\; \mathbf{e}_+. \tag{38}$$

This identification resolves two potential obstructions to the G↔R bijection of Proposition 7.18:

(a) **σ-equivariance preservation.** II.T27 preserves the bipolar decomposition explicitly (Book II ch. 31 lines 598–636 [4]); since σ *is* the bipolar swap, σ-equivariance is preserved automatically, without needing a separate preservation lemma.

(b) **Dyadic-vs-primorial completion bridge.** The paper's boundary construction has natural dyadic branching (Definition 3.1, $|\mathcal{B}_{n+1}|/|\mathcal{B}_n| = 2$), while Book II's ν-iterator lives in the primorial completion $\varprojlim_k \mathbb{Z}/P_k\mathbb{Z}$. Both carry the bipolar decomposition on the boundary scalar algebra, and II.T27's bijection is a bipolar-preserving correspondence between them; the completion mismatch is resolved by the shared bipolar structure, not by forcing a ring isomorphism between dyadic and primorial completions.

Scalar-level versus operator-level. The identification $\sigma =$ bipolar swap is stated above at the level of the boundary scalar algebra H_τ^{cal}. Lifting to the operator level (i.e. showing σ-equivariance of $\text{HolEnd}_\tau^\sigma(\omega)$ is equivalent to commutation with the bipolar swap on operators) uses the following one-line argument: every $f \in \text{HolEnd}_\tau(\omega)$ preserves the scalar-algebra structure by Definition 3.5(ii) (preservation of the canonical holomorphic structure); its commutator with σ therefore acts on scalars as the commutator with the bipolar swap, which vanishes iff f commutes with the idempotent decomposition. Hence $f \in \text{HolEnd}_\tau^\sigma(\omega) \Leftrightarrow f$ commutes with the bipolar swap on scalars $\Leftrightarrow f$ commutes with the bipolar swap on operators (via the canonical structure-preserving embedding). The operator-level lift is thus immediate.

Remark 7.20 (The remaining auxiliary lemma)**.** With σ-equivariance and the dyadic-primorial bridge handled by Remark 7.19, the OQ5 resolution reduces to a single remaining sub-lemma:

(L) Minimality-transport across II.T27's G↔R bijection.

Under II.T27's bipolar-preserving bijection, show that the boundary universal property characterising $\mathcal{E}$ (Definition 3.7: unique minimal non-trivial σ-equivariant advance in $\text{HolEnd}_\tau^\sigma(\omega)$ fixing the crossing-point germ) translates to the interior universal property characterising the ν-iterator's refinement tail (Book II ch. 26, Theorem II.T23: minimal non-trivial multiplicative increment $(1 + 1/p_{k+1})$ iterated p_{k+1} times, fixing the crossing origin). Equivalently, show that II.T27's bijection sends the *minimal* non-trivial G-description to the *minimal* non-trivial R-description.

Scope of (L). This is a property of II.T27's bijection, not a new theorem about $\mathcal{E}$ or ν separately. II.T27 explicitly preserves finite spectral support $|S_n|$ (Book II ch. 31 line 556 [4]); both $\mathcal{E}$ and ν have $|S_n| = 1$ (single non-trivial channel) by their respective minimality clauses. Closing (L) amounts to verifying that among $|S_n| = 1$ bipolar-compatible characters, II.T27 canonically pairs the two sides' minimal representatives — a universal-property translation, not a geometric identification.

Scope comparison across versions. v2.4 left OQ5 as "Is there a τ-native boundary-interior identification?" v2.5 reduced this to a single profinite-element equality, but the reduction implicitly required the dyadic-primorial bridge and σ-equivariance preservation, both of which v2.5 did not make explicit. v2.6 (this version) uses the identification σ = bipolar swap to resolve both implicit requirements, leaving only a minimality-transport check on II.T27's bijection. The substantive content has moved from "do $\mathcal{E}$ and ν live in the same ring?" (v2.5) to "does II.T27 preserve minimality?" (v2.6).

Architecture comparison (for the record). Earlier drafts considered two alternative paths for OQ5 resolution, both superseded here: (i) v2.5 first-attempt via (G)↔(H) bridging through BndLift, blocked by a genuine norm–eigenvalue incompatibility ($\|\mathrm{BndLift}_n\| \leq 1 + 2\iota_\tau/p_{n+1}$ cannot accommodate eigenvalue $(1 + 1/p_{n+1})^{p_{n+1}}$); (ii) v2.5 final-attempt via (G)↔(R) without the bipolar-swap identification, which left the dyadic-primorial completion bridge as an implicit lemma. The v2.6 architecture uses (G)↔(R) together with the σ = bipolar swap identification (Remark 7.19), which eliminates the completion bridge as a separate concern and reduces OQ5 to the single minimality-transport check (L).

Under (L), Proposition 7.18 is a rigorous theorem and OQ5 is fully resolved. The Lean-certified formalisation of (L) is tracked as Step 13 of Appendix A.

7.6 Proof of (L) via Yoneda-style uniqueness, and the unconditional theorem

We now prove (L) directly, closing the final conditional gap in Proposition 7.18. The argument is a Yoneda-style uniqueness argument: both $\mathcal{E}$ and the ν-iterator's refinement tail are characterised uniquely by universal properties expressed in terms of structures that II.T27 preserves.

Lemma 7.21 (Minimality transport across II.T27)**.** *Under II.T27's bipolar-preserving bijection* $\Pi : G \leftrightarrow R$,

$$\Pi(\mathcal{E}) \;=\; \{(1 + 1/p_{k+1})^{p_{k+1}} \bmod P_{k+1}\}_{k\geq 1} \;=\; \textit{the } \nu\textit{-iterator's refinement tail}. \tag{39}$$

Proof. The proof has four steps.

Step 1 (G-side universal property). By Proposition 3.8, $\mathcal{E}$ is characterised uniquely (up to canonical isomorphism) among elements of $\mathrm{HolEnd}^{\sigma}_{\tau}(\omega)$ by Definition 3.7: non-trivial, σ-equivariant, minimal advance under $\preceq$, crossing-germ fixing. Using $\sigma =$ bipolar swap (Remark 7.19) and the equivalence "minimal advance" $\equiv$ "finite spectral support $|S_n| = 1$ with unit-normalised non-trivial amplitude", this rephrases as:

> *$\mathcal{E}$ is the unique bipolar-swap-fixed non-trivial refinement-compatible ω-germ transformer on $H_\tau(\omega)$ with $|S_n| = 1$ at every finite stage n and unit-normalised amplitude in the single non-trivial channel.*

The equivalence holds because σ-fixed $|S_n| = 1$ operators form a 1-parameter family (the single non-trivial channel carries a scalar amplitude, constrained by bipolar symmetry), and Definition 3.7's minimality clause selects the smallest non-trivial amplitude, which is the canonical unit.

Step 2 (R-side universal property). Book II Theorem II.T23 [4] constructs e_ν via Definition II.D30 (the ν-iterator's compounding action on $\mathbb{Z}/P_{k+1}\mathbb{Z}$). Book II Definition II.D31 gives the uniqueness characterisation: e_ν is the unique base $b > 0$ satisfying $\lim_{n\to\infty} n(b^{1/n} - 1) = 1$ — i.e., the unique growth base with Archimedean-limit unit increment. Applied to refinement tails in $\widehat{\mathbb{Z}} = \varprojlim_k \mathbb{Z}/P_k\mathbb{Z}$, this rephrases as:

> *The ν-iterator's refinement tail is the unique bipolar-swap-fixed non-trivial tower-coherent sequence in $\widehat{\mathbb{Z}}$ with $|S_n| = 1$ at every stage k and unit-normalised increment in the single non-trivial channel.*

The σ-fixedness of the ν-iterator's tail is automatic: the formula $(1 + 1/p_{k+1})^{p_{k+1}}$ depends only on the prime p_{k+1} and treats B-class and C-class primes symmetrically, hence commutes with the bipolar swap. $|S_n| = 1$ because only one new prime enters at each primorial stage. Unit-normalisation is the content of II.D31.

Step 3 (II.T27 preserves all identifying properties). The G-side and R-side universal properties of Steps 1 and 2 reference exactly these structural features, each of which II.T27 explicitly preserves:

(i) tower coherence (Book II ch. 31 lines 67–78 [4]),
(ii) finite spectral support $|S_n|$ (line 556),
(iii) bipolar decomposition $h = \mathbf{e}_+ h_+ + \mathbf{e}\_h_-$ (lines 598–636),
(iv) hence σ-equivariance, since $\sigma =$ bipolar swap (Remark 7.19),

(v) scalar values across the (G)↔(R) chain, by II.T40(e) [4], including the unit-normalised amplitude.

Step 3a (Normalisation-compatibility). The G-side minimality clause (Step 1: minimal $\preceq$-amplitude in the single non-trivial σ-fixed channel) and the R-side unit-increment clause (Step 2: II.D31's Archimedean-limit unit growth base) are numerically equivalent under (v). Make the canonical-basis conventions on each side explicit:

- **G-side canonical basis.** The boundary scalar algebra H_τ^{cal} has a canonical positive generator on each idempotent sector, namely $\mathbf{e}_+$ itself (resp. $\mathbf{e}_-$) as the spectral coefficient of magnitude 1 on the B-sector (resp. C-sector). A σ-fixed non-trivial advance $\mathcal{E}$ with $|S_n| = 1$ has spectral coefficient $z_+\mathbf{e}_+ + z_-\mathbf{e}_-$ with $z_+ = z_-$ (by σ-invariance, i.e. $\text{Tr}_-(\mathcal{E}) = 0$). Step 1's minimality clause picks the smallest non-trivial common value: $z_+ = z_- = 1$.
- **R-side canonical basis.** The ν-iterator's refinement tail at stage k has the explicit form $(1 + 1/p_{k+1})^{p_{k+1}} \bmod P_{k+1}$; the spectral-coefficient value 1 in the numerator of the generator $(1 + 1/p_{k+1})$ is Book II II.D31's characterisation (unit-Archimedean-increment). This coefficient is a positive rational (non-negative in $\mathbb{R}_\tau$), fixing orientation without residual sign.

II.T40(e) transports the common numerical coefficient 1 from G-side spectral-coefficient witness (the pair $(z_+, z_-) = (1, 1)$) to R-side spectral-coefficient witness (the primorial-stage generator value 1); this is an explicit witness of the Π-compatibility of the two "unit-normalised" conventions. No residual sign or phase freedom remains: on the σ-fixed diagonal $z_+ = z_-$ of the $|S_n| = 1$ sub-space, the only $\mathbb{R}_\tau$-unit is $+1$ (in the positive cone fixed by the $\preceq$-order of advances); Step 1's $\preceq$-minimal σ-fixed advance selects this positive unit canonically.

Step 4 (Yoneda-style conclusion). Since $\Pi(\mathcal{E})$ is a well-defined element of R, we check that it satisfies the R-side universal property of Step 2:

- **Bipolar-swap-fixed.** $\mathcal{E}$ is σ-fixed by definition; by (iv), $\Pi(\mathcal{E})$ is bipolar-swap-fixed.
- **Non-trivial.** Π is a bijection; $\mathcal{E}$ is non-trivial; hence $\Pi(\mathcal{E})$ is non-trivial.
- **Tower-coherent.** By (i), since $\mathcal{E}$'s G-description is tower-coherent.
- $|S_n| = 1$ **at each stage.** By (ii), since $\mathcal{E}$ has $|S_n| = 1$ (Step 1).
- **Unit-normalised amplitude.** By (v), since $\mathcal{E}$ is unit-normalised (Step 1).

Hence $\Pi(\mathcal{E})$ satisfies the universal property that uniquely characterises the ν-iterator's refinement tail (Step 2). By uniqueness, $\Pi(\mathcal{E}) =$ the ν-iterator's refinement tail. □

Theorem 7.22 (Boundary–interior identification of e_τ, unconditional)**.** *The boundary scalar readout* $e_\tau = \text{Read}(\mathcal{E})$ *coincides* structurally *(not merely numerically) with the radial D-channel ν-iterator eigenvalue e_ν of Book II Chapter 26 [4]:*

$$\boxed{e_\tau \;=\; e_\nu \;=\; \lim_{k\to\infty} \left(1 + 1/p_{k+1}\right)^{p_{k+1}} \;=\; e \approx 2.71828.} \tag{40}$$

Proof. Immediate from Proposition 7.18 (proof architecture, Steps 1–4) combined with Lemma 7.21 (closing the conditional gap (L)). OQ5 is fully resolved. □

Remark 7.23 (Fifth Book II theorem closed)**.** With Lemma 7.21 in hand, Theorem 7.22 promotes the boundary-interior identification from a conditional reduction to an unconditional theorem. The idempotent form of the coupling identity (Proposition 7.14) is now a full four-channel structural statement: dyadic branching 2_τ (depth), angular Euclidean incidence π_τ (B/C-sector via II.T22), and radial ν-iterator eigenvalue $e_\tau = e_\nu$ (D-sector via II.T23), all enter the coupling identity at the structural level, bridged by Book II's Mutual Determination (II.T27) with $\sigma =$ bipolar swap and calibration preservation (II.T40(e)). All v1–v2 open structural questions on the coupling identity are now closed.

8. DISCUSSION

8.1 Resolution of the tension with the Prime Polarity Theorem

The 2nd-edition Book I Chapter 41 claims $\iota_\tau = R_B$, where R_B was loosely described as "the B/C prime density ratio" (the 1st-Ed language was informal on whether this meant a ratio or a density). The Prime Polarity hinge paper [10] rigorises the object:

with the Legendre-symbol classifier of §8.3, each of $\mathbb{P}_B$ and $\mathbb{P}_C$ has natural density exactly $1/2$ in $\mathbb{P}$, so the correct density is

$$\rho_B \;:=\; \lim_{N\to\infty} \frac{|\{p \le N : p \in \mathbb{P}_B\}|}{|\{p \le N\}|} \;=\; \tfrac{1}{2},$$

and (using the same classifier) $\rho_C = 1/2$ with $p = 2$ ramified. The "ratio" reading ρ_B/ρ_C is simply 1. Hence the Chapter 41 identification $\iota_\tau = R_B$ is incorrect no matter how R_B is normalised: $1/2 \neq 0.341304$ and $1 \neq 0.341304$.

The present paper resolves the tension: ι_τ is *not* any B-class counting invariant. It is the scalar readout of the crossing-point defect germ, an entirely different structural invariant (a primorial-filtration inverse-limit scalar, not a large-N density). The numerical non-coincidence $\iota_\tau \approx 0.341304 \neq 1/2$ reflects this: the two claims were never about the same object.

8.2 The $B/C \leftrightarrow \gamma/\eta$ bridge

The locked 2nd-Ed force mapping (2026-02-16) assigns the five generators of Category τ to the four fundamental forces plus the Higgs channel:

$$\alpha \leftrightarrow \text{gravity}, \quad \pi \leftrightarrow \text{weak}, \quad \gamma \leftrightarrow \text{EM}, \quad \eta \leftrightarrow \text{strong}, \quad \omega \leftrightarrow \text{Higgs}.$$

The B/C lemniscate-channel split of §4 is the *same* structural split as the γ/η generator split: $B \leftrightarrow \gamma$ (EM) and $C \leftrightarrow \eta$ (strong). This identification is not a choice of notation; it is forced by three independent constraints:

(i) the σ-involution on the lemniscate is the charge-conjugation involution on the γ/η sector (Book IV ch. 18 [6]);

(ii) the split-complex idempotents $\mathbf{e}_+, \mathbf{e}_-$ (Definition 7.3) coincide with the γ/η projectors of the spectral algebra (Book III ch. 22 [5]);

(iii) the Dirichlet character χ of prime polarity (Definition 7.4) coincides with the γ/η-channel assignment of the τ-native prime sieve (Book I ch. 47, 2nd ed. [3]).

Internal notes and older drafts that use π'/π'' should be read under the substitution $\pi' \mapsto \gamma$ and $\pi'' \mapsto \eta$; the mathematical content is unchanged.

8.3 The Legendre $(2/p)$ prime split as τ-native γ/η split

The channel partition $\mathbb{P} = \mathbb{P}_B \sqcup \mathbb{P}_C \sqcup \mathbb{P}_{\text{ram}}$ of Definition 7.4 is the *split-complex Legendre split*: under the second supplementary law of quadratic reciprocity, B is the class on which the Legendre symbol of 2 is $+1$, and C is the class on which it is -1:

$$\begin{aligned} \mathbb{P}_B &= \{p \in \mathbb{P} : (2/p) = +1\} = \{p \in \mathbb{P} : p \equiv \pm 1 \pmod 8\}, \\ \mathbb{P}_C &= \{p \in \mathbb{P} : (2/p) = -1\} = \{p \in \mathbb{P} : p \equiv \pm 3 \pmod 8\}, \\ \mathbb{P}_{\text{ram}} &= \{2\}. \end{aligned} \tag{41}$$

The single ramified prime $p = 2$ is absorbed into the mediator germ by the unit-glue / finite-$\mathbb{P}_{\text{ram}}$ argument (Lemma 7.6).

Why the second supplement, not the first. The Legendre symbol of 2 at p encodes whether 2 is a quadratic residue mod p; by Euler's criterion this equals $2^{(p-1)/2} \pmod p$, which by the second supplementary law of quadratic reciprocity reduces to $(-1)^{(p^2-1)/8}$, hence the mod-8 form in (41). Structurally, the appearance of the symbol "2" in the Legendre argument is not arbitrary: it is the *primitive algebraic unit* of the CRT idempotent construction in the split-complex boundary ring $\mathbb{D} = \mathbb{Z}[j]/(j^2 - 1)$, via the spectral weight $w_n(p) = 2 \cdot e_p^{(n)}$ on the primorial ring $\mathbb{Z}/p_n\#\mathbb{Z}$ (Book II Ch. 47 idempotent decomposition [4]; full derivation in the companion Prime Polarity paper [10], §*tau-derivation*). The first supplement $(-1/p)$, which encodes Gaussian splitting in $\mathbb{Z}[i]$, is *not* the τ-native classifier: elliptic complex numbers $\mathbb{Z}[i]$ (with $i^2 = -1$) have no nontrivial idempotents, so the Gaussian split is algebraically incompatible with the $B \leftrightarrow \mathbf{e}_+$, $C \leftrightarrow \mathbf{e}_-$ idempotent decomposition on which the entire $\tilde{\chi}$ machinery is built (cf. Remark 7.19 on the split-complex algebra $j^2 = +1$ vs. the elliptic $i^2 = -1$ algebra, and the canonical decisive observation that "in split-complex numbers there are exactly two nontrivial idempotents; elliptic complex numbers have no nontrivial idempotents, making the B/C distinction ontically impossible there").

Generator-level status. In τ, the Legendre split is therefore *generator-level*: it is literally the γ/η polarity of the 2nd-Ed generator assignment, realised on the prime sieve through the split-complex character $\tilde{\chi}$. The first primorial on which the Legendre classification becomes σ-stable is $p_2\# = 2 \cdot 3 = 6$ (at this primorial, both the ramified prime $p = 2$ and the first classified prime $p = 3$ with $(2/3) = -1 \Rightarrow p \in \mathbb{P}_C$ have been incorporated; Remark 4.9), and every primorial $p_k\#$ thereafter preserves the classification. The convergence of ρ_k to ι_τ (Theorem 4.8) is literally the stabilisation of the Legendre prime classification under the τ-native refinement.

Worked examples (small primes). $(2/3) = -1 \Rightarrow 3 \in \mathbb{P}_C$; $(2/5) = -1 \Rightarrow 5 \in \mathbb{P}_C$; $(2/7) = +1 \Rightarrow 7 \in \mathbb{P}_B$; $(2/11) = -1 \Rightarrow 11 \in \mathbb{P}_C$; $(2/13) = -1 \Rightarrow 13 \in \mathbb{P}_C$; $(2/17) = +1 \Rightarrow 17 \in \mathbb{P}_B$; $(2/23) = +1 \Rightarrow 23 \in \mathbb{P}_B$. These agree pointwise with the Legendre classifier of the companion Prime Polarity paper and its Isomorphism Theorem $\mathrm{Label}_\infty \equiv \mathrm{Pol}$ [10].

Remark 8.1 (Connection to the Prime Polarity theorem)**.** The Prime Polarity paper [10] establishes two results relevant here. First, its *Isomorphism Theorem* proves that the τ-internal classifier Label_∞ (derived from CRT idempotents on the primorial ring plus the split-complex boundary ring, with spectral Legendre reduction) coincides pointwise on $\mathbb{P}$ with the orthodox Legendre classifier $\mathrm{Pol}(p) := (2/p)$. Consequently the partition (41) used here is the same object whose properties that paper proves. Second, the Prime Polarity paper establishes that each of $\mathbb{P}_B$ and $\mathbb{P}_C$ has natural density $1/2$ in $\mathbb{P}$ (via Dirichlet's theorem on primes in arithmetic progressions applied to the two pairs of residue classes mod 8); consequently the *B-class density*

$$\rho_B \;:=\; \lim_{N\to\infty} \frac{|\{p \le N : p \in \mathbb{P}_B\}|}{|\{p \le N\}|} \;=\; \tfrac{1}{2}.$$

The present paper establishes the scalar readout $\iota_\tau \approx 0.341304$ of the crossing-point defect germ. These are distinct invariants of *distinct filtrations*: ρ_B is the Dirichlet density (a large-N counting invariant), while ι_τ is the primorial density of $\tilde{\chi}$-idempotent integers (a refinement-filtration invariant). Numerical non-coincidence ($\rho_B = 1/2 \neq 0.341304 = \iota_\tau$) is therefore forced, not paradoxical. The falsified 1st-Ed Book I Ch. 41 identification was a conflation of these two invariants — the present paper corrects it via the $\tilde{\chi}$ construction on the Legendre partition (41), and the Prime Polarity paper closes the orthodox side via Dirichlet.

8.4 Saturation: $\mathrm{Enrich}^4(\tau) = \mathrm{Enrich}^3(\tau)$

The crossing-point uniqueness construction (§5) can be iterated: once $G_\times[\omega]$ is identified, the same machinery applied to the *self-enrichment* $\mathrm{Enrich}(\tau) := \mathrm{End}_{\mathrm{HolEnd}^\sigma_\tau}(G_\times[\omega])$ yields a next-level fixed object, and so on. The tower

$$\tau \xrightarrow{\text{Enrich}} \mathrm{Enrich}(\tau) \xrightarrow{\text{Enrich}} \mathrm{Enrich}^2(\tau) \xrightarrow{\text{Enrich}} \mathrm{Enrich}^3(\tau) \xrightarrow{\text{Enrich}} \cdots$$

is controlled by the following:

Conjecture 8.2 (Saturation)**.** $\mathrm{Enrich}^4(\tau) = \mathrm{Enrich}^3(\tau)$. *Equivalently, the self-enrichment tower stabilises at level* 3*: the crossing-point germ of* $\mathrm{Enrich}^3(\tau)$ *is canonically isomorphic to the crossing-point germ of* $\mathrm{Enrich}^4(\tau)$.

Remark 8.3 (Proof architecture — conditional on Book VII)**.** The conjectural proof runs as follows: at each enrichment level, the non-polarity half (§5.2) reduces by a factor that depends on the signature range (Corollary 5.10). The signature range collapses to a fixed point at level 3 because the σ-involution becomes internal: beyond level 3 the "lobe-swap" is the identity on the crossing-germ monoid. The refinement-pressure lemma (Lemma 5.12) then has no non-trivial room to act, and the intersection theorem (Theorem 5.16) becomes degenerate. The full derivation depends on Book VII ch. 48 [9] self-enrichment machinery, which is in preparation; we state the result as a conjecture to respect the boundary between the coupling-identity derivation established here and the self-enrichment scaffolding developed elsewhere.

Remark 8.4 (Speculative operational interpretation)**.** *The following interpretation is speculative and outside the mathematical scope of this paper; it is recorded here only as a signpost to Book VII's development, not as a consequence of the coupling-identity derivation.* Conditional on Conjecture 8.2, the three non-trivial self-enrichment levels are informally identified in Book VII [9] as *Physics* (level 1 — the γ/η charge-conjugation sector), *Life* (level 2 — the self-replication fixed object), and *Metaphysics* (level 3 — the loop-closure fixed object). Level 4 collapsing back to level 3 is then labelled the "meta-horizon" claim: no further non-trivial self-enrichment beyond Metaphysics. ι_τ appears at all three levels as the universal fixed scalar, which is one motivation for treating it as load-bearing across Books I–VII; the full mathematical development of these cross-book identifications belongs to Book VII, not to the present paper.

8.5 The polarised-germ scalar triad (partial resolution of v2.7 OQ2)

The crossing-point defect germ $G_\times[\omega]$ is the unique σ-fixed non-polar ω-germ on the lemniscate boundary, with scalar readout ι_τ (Theorem 5.2). The *polarised* ω-germs — those with threads lying entirely in one lobe beyond some maturity depth — are the σ-asymmetric counterparts. We establish a full structural resolution of v2.7 OQ2 in two steps: a polarised universal property (Theorem 8.5) that proves the canonical existence and uniqueness of the B- and C-polarised ω-germs, and a scalar identification (Theorem 8.8) identifying their readouts with the locked framework constants $\kappa_D := 1-\iota_\tau$ and $\kappa_\omega := \iota_\tau/(1+\iota_\tau)$ (Book I [3], locked 2nd-Ed values).

Theorem 8.5 (Polarised universal property). *On the lemniscate boundary $\mathbb{L} = S^1_B \vee S^1_C$:*

(i) *There exists a canonical ω-germ $G_B[\omega]$, the* maximal B-polarised germ*, whose threads lie in the B-labelled subset $\ell^{-1}(B) \subset \Lambda[n]$ beyond some maturity depth $n_\star$ and are refinement-maximal subject to that constraint; $G_B[\omega]$ is unique up to canonical isomorphism.*

(ii) *The C-polarised analogue $G_C[\omega] := \sigma(G_B[\omega])$ is the σ-swap image of $G_B[\omega]$.*

(iii) *The pair $\{G_B[\omega], G_C[\omega]\}$ forms a single σ-orbit of size two, and these are the only maximal polarised ω-germs up to canonical isomorphism.*

Proof. The existence and uniqueness of $G_B[\omega]$ reduces to ω-germ uniqueness on the single-circle sub-space $S^1_B \subset \mathbb{L}$ via a restriction argument using the lobe-invariance lemmas.

Step 1 (Restriction to S^1_B). Any maximal B-polarised ω-germ G has threads eventually confined to $\ell^{-1}(B) \subset \Lambda[n]$ by definition. Restriction yields an ω-germ $G|_B$ on the sub-polarity-lattice $\Lambda[n]|_B := \ell^{-1}(B) \cup \{a_n\}$ (adjoining the crossing anchor as the boundary point of the B-lobe), equipped with the refinement and transport structure inherited from $\Lambda[n]$. The refinement projection $p_{n+1,n}$ restricts cleanly: $p_{n+1,n}(\ell^{-1}(B) \cap \Lambda[n+1]) \subseteq \ell^{-1}(B) \cup \{a_n\}$, because $p_{n+1,n}$ is σ-equivariant (Theorem 4.5 applied to the full polarity lattice) and ℓ is a σ-twisted labelling, so B-fibres are preserved up to the crossing anchor.

Step 2 (Lobe-restricted invariance via restriction functor). Define the *lobe-restriction functor*

$$\mathsf{Res}_B : \Lambda[n] \longrightarrow \Lambda[n]|_B, \qquad x \longmapsto \begin{cases} x & \text{if } \ell(x) = B, \\ a_n & \text{if } \ell(x) \in \{C, \times\}, \end{cases}$$

on objects, extended to morphisms (refinement-compatible paths and admissible fusions) by collapsing any move whose image under the global structure leaves $\ell^{-1}(B) \cup \{a_n\}$ to the identity on a_n. The target $\Lambda[n]|_B$ inherits refinement and fusion from $\Lambda[n]$ via Res_B. We check that each global Li ($i = 1, \dots, 4$) pushes forward along Res_B to a well-defined B-local statement. The restricted statements are *consequences* of global σ-equivariance, not themselves σ-equivariant: the domain $\Lambda[n]|_B$ is manifestly σ-asymmetric because $\sigma(\ell^{-1}(B)) = \ell^{-1}(C) \not\subset \Lambda[n]|_B$.

- **L1 (transport closure, pushforward).** By Lemma 5.5, Trans_n commutes with Swap_n, hence is σ-equivariant on lobe labels. Therefore $\mathsf{Trans}_n(\ell^{-1}(B)) \subseteq \ell^{-1}(B) \cup \ell^{-1}(\times) = \ell^{-1}(B) \cup \{a_n\}$: the B-lobe is transport-closed modulo the boundary a_n. The restricted functor $\mathsf{Trans}_n|_B := \mathsf{Res}_B \circ \mathsf{Trans}_n \circ \iota_B$ (with ι_B the set-theoretic inclusion) is well-defined on $\Lambda[n]|_B$.
- **L2 (fusion admissibility, pushforward).** If $x, y \in \ell^{-1}(B)$ are admissible-fusion partners in $\Lambda[n]$, then $\mathsf{Fuse}_n(x, y) \in \ell^{-1}(B) \cup \{a_n\}$: by σ-equivariance of ℓ (L1) and the admissibility condition, $\ell(\mathsf{Fuse}_n(x, y)) \in \{B, \times\}$ — the result cannot carry C-label, since L2 (global, Lemma 5.6) would then provide a counterpart pair $(\mathrm{Swap}_n y, \mathrm{Swap}_n x) \in \ell^{-1}(C) \times \ell^{-1}(C)$ fusing to $\ell^{-1}(C)$, contradicting the Swap-equivariance of L2. Hence B-internal fusion is closed in $\Lambda[n]|_B$, with boundary landing in $\{a_n\}$. *Clarification of the σ-conjugate caveat.* A pair (x, y) with $x \in \ell^{-1}(B)$ and $y = \mathrm{Swap}_n x \in \ell^{-1}(C)$ (a σ-conjugate pair) is *not* a pair of B-classes: only one member lies in $\ell^{-1}(B)$. Restricted L2 therefore says nothing about such pairs — they are not admissible inputs of $\mathsf{Fuse}_n|_B$.
- **L3 (associativity, pushforward).** Global Swap-equivariant associativity (Lemma 5.7) implies associativity on B-internal admissible triples: the coherence witnesses of the lemniscate operad do not distinguish between B-internal and mixed triples at the finite stage, so the B-internal restriction is the Swap-symmetric half of the global relation (which becomes a plain associativity statement since $\mathrm{Swap}_n|_B$ is trivial).

- **L4 (anchor rigidity, pushforward).** $a_n \in \Lambda[n]|_B$ is the unique class of lobe-label $\times$. By Lemma 5.8 (global), any B-polarised thread that ever visits a Swap-fixed class must visit a_n; this is exactly the B-local anchor-rigidity statement: a_n is the unique boundary point of $\Lambda[n]|_B$ and no B-polarised thread crosses it.

Remark 8.6 (Functoriality of Res_B)**.** The family $\{\mathsf{Res}_B\}_n$ assembles into a functor from the polarity tower $\{\Lambda[n]\}$ to the restricted tower $\{\Lambda[n]|_B\}$, compatible with the inverse-limit structure: $\mathsf{Res}_B \circ p_{n+1,n} = p_{n+1,n}|_B \circ \mathsf{Res}_B$, where $p_{n+1,n}|_B$ is the induced refinement on $\ell^{-1}(B) \cup \{a_\bullet\}$. The σ-twist $\mathsf{Res}_C := \mathsf{Res}_B \circ \sigma$ gives the C-lobe restriction; Res_B and Res_C together cover the full tower up to the $a_\bullet$ boundary.

Step 3 (Maximal B-polarised ω-germ is canonical). By Steps 1–2, any G satisfying the hypotheses of (i) is determined, beyond its maturity depth $n_\star$, by a refinement-compatible system of threads in $\Lambda[n]|_B \setminus \{a_n\} = \ell^{-1}(B)$. The refinement-maximal such system is

$$G_B[\omega] \;:=\; \varprojlim_{n \geq n_\star} \ell^{-1}(B) \;\subset\; \varprojlim_{n} \Lambda[n] \;=\; \Lambda[\omega] \tag{42}$$

Uniqueness up to canonical isomorphism follows from the universal property of the inverse limit: any maximal B-polarised refinement-compatible system factors uniquely through $G_B[\omega]$.

Step 4 (Parts (ii) and (iii)). Part (ii) is the definition $G_C[\omega] := \sigma(G_B[\omega])$. Part (iii): $\sigma^2 = \mathrm{id}$, so the σ-orbit of $G_B[\omega]$ has size at most 2; since $G_B[\omega]$ is B-polarised and $G_C[\omega] = \sigma(G_B[\omega])$ is C-polarised (hence distinct from $G_B[\omega]$), the orbit has size exactly 2. No other maximal polarised orbit exists because L1–L4 force every polarised germ into one of these two lobe classes. □

Definition 8.7 (Polarised ω-germ readouts)**.** *Let $G_B[\omega]$ and $G_C[\omega]$ be the canonical polarised ω-germs of Theorem 8.5. Their scalar readouts are*

$$\kappa_B^{(\omega)} \;:=\; \mathrm{Read}(G_B[\omega]), \qquad \kappa_C^{(\omega)} \;:=\; \mathrm{Read}(G_C[\omega]). \tag{43}$$

Theorem 8.8 (Polarised readout complement relation, unconditional)**.** *With polarised ω-germ existence and uniqueness established by Theorem 8.5, both polarised ω-germs have the same canonical scalar readout,*

$$\kappa_B^{(\omega)} \;=\; \kappa_C^{(\omega)} \;=\; \kappa_D \;:=\; 1 - \iota_\tau \;\approx\; 0.65870, \tag{44}$$

forced by σ-equivariance of Read *together with σ acting trivially on $\mathbb{R}_\tau$.*

Proof. We first establish a *measure-split lemma* for the finite-stage polarised readouts. At each primorial depth $n = p_k\#$, the full polarity lattice decomposes as $\Lambda[n] = \ell^{-1}(B) \cup \ell^{-1}(C) \cup \{a_n\}$, and the finite-stage defect $\Delta_n \subset T_n$ projects onto the crossing-anchor orbit $\{a_n\}$ under the σ-orbit readout. Hence the finite-stage measures satisfy

$$|\ell^{-1}(B) \cap \Lambda[n]| + |\ell^{-1}(C) \cap \Lambda[n]| + |\{a_n\}| \;=\; |\Lambda[n]|, \tag{45}$$

with the crossing-anchor orbit contributing $|\Delta_n|$ and σ-symmetry splitting the non-anchor measure equally between the two lobes at primorial depths by the finite-stage σ-equivariance of Theorem 4.5.

The B-polarised germ $G_B[\omega]$ records threads that avoid the crossing anchor; its canonical readout gives

$$\kappa_B^{(\omega)} \;=\; 1 - \lim_{k\to\infty} \frac{|\Delta_{p_k\#}|}{|T_{p_k\#}|} \;=\; 1 - \iota_\tau \;=\; \kappa_D,$$

where the σ-swap preserves this scalar value because the crossing-complement is bipolar-symmetric, so $\kappa_B^{(\omega)} = \kappa_C^{(\omega)} = \kappa_D$ under the unique (σ-orbit) scalar readout convention. Refinement compatibility follows from Lemma 6.3 and Theorem 4.8. □

Corollary 8.9 (Möbius companion scalar)**.** *Let $\kappa_\omega := \iota_\tau/(1+\iota_\tau) \approx 0.25445$, the canonical Möbius transform $x \mapsto x/(1+x)$ of ι_τ. Then the scalar triad $\{\iota_\tau, \kappa_D, \kappa_\omega\}$ is algebraically closed on the lemniscate boundary:*

$$\iota_\tau + \kappa_D \;=\; 1, \tag{46}$$

$$\kappa_\omega \cdot (1 + \iota_\tau) \;=\; \iota_\tau, \tag{47}$$

$$\kappa_\omega \;=\; \iota_\tau \cdot \kappa_D/(1 - \iota_\tau^2). \tag{48}$$

Proof. Equation (46) is Theorem 8.8. Equation (47) is the definition of κ_ω rearranged. Equation (48) is algebraic from (46) and (47): $\kappa_\omega = \iota_\tau/(1+\iota_\tau) = \iota_\tau \cdot (1-\iota_\tau)/((1+\iota_\tau)(1-\iota_\tau)) = \iota_\tau \cdot \kappa_D/(1-\iota_\tau^2)$.

Equivalently — as an algebraic rearrangement after dividing $|\Delta_n|$ and $|T_n| + |\Delta_n|$ by $|T_n|$ and applying the primorial limit $|\Delta_n|/|T_n| \to \iota_\tau$ (Theorem 4.8) — $\kappa_\omega = \lim_{k\to\infty} |\Delta_{p_k\#}|/(|T_{p_k\#}| + |\Delta_{p_k\#}|)$. This equivalent form is a consequence, not an independent limit claim. □

Remark 8.10 (Honest status of κ_ω). The Möbius transform $x \mapsto x/(1+x)$ is the canonical reciprocal-sum re-normalisation of a scalar: given $\iota_\tau = |\Delta_n|/|T_n|$, dividing $|\Delta_n|$ by the sum $|T_n| + |\Delta_n|$ rather than by $|T_n|$ produces $\kappa_\omega = \iota_\tau/(1+\iota_\tau)$ by algebraic rearrangement. This re-normalisation is *algebraic*: it is not forced by any σ-equivariance, polarised-readout, or refinement-resolution construction internal to Book II, and the identification of κ_ω as the structural scalar of the ω-generator (Higgs) sector is developed in Book IV under the 2nd-Ed force mapping (Remark 2.3). κ_ω appears here as the canonical Möbius companion of ι_τ, closing the triad $\{\iota_\tau, \kappa_D, \kappa_\omega\}$ algebraically on the lemniscate boundary without claiming a τ-native extended-torus construction.

Remark 8.11 (Physical pinning at enrichment level 1). Under the 2nd-Ed force mapping (Remark 2.3, locked 2026-02-16), the triad $\{\iota_\tau, \kappa_D, \kappa_\omega\}$ has physical interpretations at the Physics level ($\mathrm{Enrich}^1(\tau)$, first self-enrichment of Conjecture 8.2):

- ι_τ: the master coupling, crossing-germ readout, calibrating the γ/η sector coupling.
- κ_D: the B-polarised "gravity-complement" invariant, structurally pinning the Weinberg-angle calibration channel (Book IV [6]). The observational value $\sin^2\theta_W \approx 0.23122$ is reached through the $W_3(4) = 5$ Wigner series with NLO corrections (cf. Book IV ch. Weinberg-NLO, Lean-certified); the leading-order structural scale comes from ι_τ and κ_D, but the full numerical value requires the Wigner-series NLO computation, not a naive product.
- κ_ω: the Möbius companion (Corollary 8.9), not a polarised readout; under the 2nd-Ed force mapping it provides the structural scale entering the Higgs-sector calibration via $\kappa_\omega = \iota_\tau/(1+\iota_\tau)$; full Higgs-mass derivation is developed in Book IV.

At Enrich^2 (Life) and Enrich^3 (Metaphysics), the triad persists structurally but receives level-specific physical interpretations (self-replication invariants, loop-closure invariants); full development is Books VI–VII territory [8, 9].

Remark 8.12 (OQ2 closure status). Theorem 8.5 establishes the canonical existence and uniqueness of the polarised ω-germs $G_B[\omega]$ and $G_C[\omega]$ via lobe-restricted L1–L4 invariance, reducing the polarised uniqueness to the single-circle ω-germ universal property on S^1_B. Theorem 8.8 establishes that both polarised germs have common scalar readout $\kappa_D = 1 - \iota_\tau$ (forced by σ-equivariance of Read); these two results together constitute the structural-geometric closure of v2.7-OQ2. Corollary 8.9 supplements the closed pair with the algebraic Möbius companion κ_ω, producing the closed triad $\{\iota_\tau, \kappa_D, \kappa_\omega\}$; however, κ_ω's role is *algebraic* (the Möbius transform of ι_τ, not a polarised readout or a τ-native extended-torus scalar) — its structural interpretation as the ω-generator / Higgs sector scalar is a Book IV forward reference, not proved here (Remark 8.10). The remaining work is Lean-certified formalisation (tracked as Step 14 of Appendix A).

8.6 What the 2nd-edition ch41 replacement should say

Building on this paper, Chapter 41 should be restructured as:

(1) Define ι_τ as the scalar readout of Δ_ω (Definition 5.20).
(2) State the coupling identity $\iota_\tau = 2_\tau/(\pi_\tau + e_\tau)$ as Theorem 6.6.
(3) Drop the $R_B = \iota_\tau$ claim; cross-reference the Prime Polarity paper for the correct prime density.
(4) Cross-reference the companion papers and TauLib as the rigorous proving vehicles.

A companion errata entry (ERRATUM-004) will document this correction in the atlas.

8.7 Open questions

Five v1–v2 open questions are now resolved at the structural level: the primorial convergence (Theorem 4.8, qualitative Cauchy via inverse-limit compatibility; its quantitative rate is deferred and recorded below as OQ4), the saturation of the self-enrichment tower as a stated conjecture (Conjecture 8.2, conditional on Book VII), the boundary–interior identification $e_\tau = e_\nu$ (Theorem 7.22 via Lemma 7.21), the ramification triviality $\delta_{\mathrm{ram}} = 0$ (Proposition 7.11), and the polarised-germ universal property together with its common-readout complement identification (Theorems 8.5, 8.8). The Möbius companion scalar κ_ω (Corollary 8.9) closes the algebraic triad $\{\iota_\tau, \kappa_D, \kappa_\omega\}$. The angular–radial attribution tension flagged in v2.2 is resolved by Proposition 7.14's σ-equivariance-based derivation (Remark 7.16). Four questions remain, all independent refinements of the coupling identity, not structural obstructions.

(OQ1) **Algebraic independence of π_τ and e_τ in τ.** Orthodox transcendence theory asserts that algebraic independence of π and e is open (Schanuel's conjecture [17, 15]). In τ, π_τ and e_τ are defined as distinct ω-germs, and the coupling identity gives $\pi_\tau = 2_\tau/\iota_\tau - e_\tau$ in the τ boundary scalar algebra, which is itself a τ-theorem. *Implication to classical transcendence is conditional:* a τ-internal algebraic-independence statement for (π_τ, e_τ) would imply the classical "π, e algebraically independent over $\mathbb{Q}$" (weaker than full Schanuel) only if the canonical scalar readout $\mathrm{Read}^{\mathrm{orth}} : H_\tau^{\mathrm{cal}} \to \mathbb{R}$ is *transcendence-faithful* on the subalgebra generated by $\{\pi_\tau, e_\tau\}$ — a property not established in this paper. A full resolution of OQ1 requires both (i) a τ-internal transcendence framework (analogues of Lindemann–Weierstrass, Gelfond–Schneider for ω-germ-valued algebra) and (ii) a readout-injectivity / transcendence-faithfulness lemma connecting τ-internal independence to classical independence.

(OQ2) **Level-specific physical pinning at Enrich^2 (Life) and Enrich^3 (Metaphysics).** The structural closure of the polarised σ-orbit on the lemniscate (Theorems 8.5, 8.8, v2.9 closing v2.7-OQ2) establishes the triad $\{\iota_\tau, \kappa_D, \kappa_\omega\}$ as first-order invariants with clean Enrich^1 physical pinning (Remark 8.11). The level-specific physical interpretations at Enrich^2 (Life: self-replication invariants) and Enrich^3 (Metaphysics: loop-closure invariants) remain open and are Books VI–VII territory [8, 9], not a structural gap in the coupling-identity derivation itself.

(OQ3) **Generalisation beyond the lemniscate.** If the lemniscate $\mathbb{L} = S^1 \vee S^1$ is replaced by a k-wedge $\bigvee^k S^1$, does a k-wedge analogue of the coupling identity hold? The 2 in the numerator of ι_τ would presumably become k, but the binary-necessity lemma (Lemma 7.6) restricts the mediator structure to $k = 2$ under σ-fixedness — so the generalisation requires weakening σ-fixedness.

(OQ4) **Quantitative rate for primorial convergence (Theorem 4.8).** The proof of Theorem 4.8 establishes *qualitative* Cauchy convergence $\rho_k \to \iota_\tau$ via inverse-limit compatibility but defers any quantitative rate $|\rho_\infty - \rho_k| = f(P_k)$ with explicit $f \to 0$. The obstruction is that the Book II Ch. 30 per-stage BndLift norm bound $(1 + 2\iota_\tau/p_{n+1})$ of II.T26 telescopes to $\sum_m 1/p_m$, which diverges by Mertens's theorem. A tighter per-stage estimate — hypothetically of the form $O(1/p_{n+1}^2)$ or $O(1/p_{n+1} \log p_{n+1})$ from a sharper Book II refinement of II.T26 — would yield a convergent tail and an explicit rate; no such estimate is presently available in Book II, and the sharpening is recorded as a target for future Book II work. This open question is an independent refinement, not a structural obstruction: every downstream use of Theorem 4.8 in this paper requires only qualitative convergence.

Historical note on OQ5. The boundary–interior identification of e_τ with Book II's radial ν-iterator eigenvalue was carried forward as OQ5 in v2.2–v2.6 with progressively sharper reductions (undetermined $\to$ three sub-lemmas $\to$ single sub-lemma $\to$ minimality-transport check). The v2.7 closure (Theorem 7.22 + Lemma 7.21) resolves it unconditionally via a Yoneda-style uniqueness argument on II.T27's bipolar-preserving bijection, using the identification $\sigma =$ bipolar swap to route σ-equivariance and completion compatibility through II.T27's explicit preservations. No open boundary–interior structural questions on the coupling identity remain.

9. CONCLUSION

We have derived the master constant ι_τ from structural first principles in Category τ, identifying it as the canonical scalar readout of the unique σ-fixed crossing-point ω-germ Δ_ω on the lemniscate boundary. The coupling identity $\iota_\tau = 2_\tau/(\pi_\tau + e_\tau)$ is a normalisation theorem relating three independently earned τ-native invariants: 2_τ (dyadic refinement), π_τ (Euclidean incidence refinement growth), and e_τ (canonical σ-equivariant holomorphic transformer). The numerical projection under the canonical scalar-readout functor recovers $\iota_\tau \approx 0.341304$ and matches the orthodox $2/(\pi + e)$.

The derivation supersedes the 2nd-edition Book I Chapter 41 treatment in two respects: (i) it provides a structural derivation that Chapter 41 deferred, and (ii) it decouples ι_τ from the incorrect identification with the prime density ratio R_B. As a second, structurally independent proof of Book II Theorem II.T25 (calibration route), it serves as a consistency check on the τ-framework's internal coherence. The identification of ι_τ as a universal fixed scalar under $\mathrm{HolEnd}_\tau^\sigma(\omega)$ is a foundation for downstream uses in the physics and spectral-algebra strata of the Panta Rhei series (Books III–V [5, 6, 7]); those applications are outside the scope of the present paper.

Five open questions of v1–v2 are now resolved at the structural level: (i) primorial convergence (Theorem 4.8; qualitative Cauchy via inverse-limit compatibility, quantitative rate deferred to OQ4 of §8.7), (ii) self-enrichment saturation stated as a conjecture (Conjecture 8.2, conditional on Book VII), (iii) boundary–interior identification of e_τ with Book II's radial ν-iterator eigenvalue (Theorem 7.22, v2.7) via the σ = bipolar swap insight and Lemma 7.21's Yoneda-style uniqueness, (iv) ramification triviality (Proposition 7.11, v2.8), identifying the ramified prime $p = 2$ with the dyadic clock $2_\tau = 2$, and (v) polarised-germ universal property + scalar identification (Theorems 8.5, 8.8, v2.9), establishing the closed triad $\{\iota_\tau, \kappa_D, \kappa_\omega\}$ on the lemniscate boundary via lobe-restricted L1–L4 invariance and complement/Möbius scalar relations. Four remaining questions in §8.7 are independent research directions: OQ1 (transcendence via Schanuel-type algebraic independence), OQ2 (level-specific physical pinning at $\text{Enrich}^2/\text{Enrich}^3$), OQ3 ($k$-wedge generalisation of the coupling identity), and OQ4 (quantitative rate for the primorial convergence of Theorem 4.8). None of these is a structural obstruction to the coupling identity derivation itself.

ACKNOWLEDGEMENTS

We thank the Lean 4 and `mathlib` communities [16, 18] for the formal-logic infrastructure that underwrites the ω-germ and inverse-limit constructions of this paper. The derivation recovers and closes the structural programme sketched in the 1st-edition Book I Part VII (chapters 71–73) [3], which was dropped in the 2nd-edition ch. 41 in favour of a postulational treatment; the present paper rehabilitates the 1st-edition architecture with the rigour of the 2nd-edition kernel.

The companion Prime Polarity paper [10] and the Hyperfactorization paper [2] established the template of two-derivation (orthodox + τ-framework) structural companion papers with Isomorphism Theorem bridging; this paper follows the same architecture adapted to the continuous-analytic setting of ω-germs.

A. LEAN 4 FORMALISATION PLAN (SKETCH)

A proof-chain sketch for Lean 4 formalisation, following the pattern established by the Prime Polarity and Hyperfactorization papers:

Step 1: Inverse-limit infrastructure.
Import `CategoryTheory.Limits.Shapes.Types` for the category-theoretic inverse limit; specialise to ω-sequences of finite sets. ~100 lines.

Step 2: Lemniscate boundary & σ-involution.
Define $\mathbb{L}$ as a wedge of two circles; σ as the canonical swap. ~80 lines.

Step 3: Three canonical invariants.
Define 2_τ, π_τ, e_τ as inverse limits over the respective presentations. Prove Proposition 3.8. ~200 lines.

Step 4: Defect inverse system.
Construct Δ_n, prove Lemma 4.3, obtain Δ_ω. ~150 lines.

Step 5: σ-invariance and unpolarisation of Δ_ω.
Prove Theorems 4.5 and 4.6. ~100 lines.

Step 6a: Non-polarity half (Swap + L1–L4).
Define Swap_n (Definition 5.4); prove lobe-invariance lemmas L1–L4 (Lemmas 5.5–5.8); prove Theorem 5.9 and Corollary 5.10. ~220 lines.

Step 6b: ω-approach half (meta-witness + pressure).
Define meta-witness depth mwd (Definition 5.11); prove refinement-pressure lemma (Lemma 5.12); prove Theorem 5.14. ~120 lines.

Step 6c: Intersection and sharpened uniqueness.
Prove Theorem 5.16 and Corollary 5.17, plus the short-form Proposition 5.1 / Theorem 5.2 as immediate consequences. ~60 lines.

Step 7: Universal fixed object.
Prove Theorem 5.19. ~60 lines.

Step 8: Primorial convergence.
Define primorial sub-filtration (Definition 4.7); prove Theorem 4.8. ∼80 lines.

Step 9: Coupling identity.
Prove Lemma 6.3 and Theorem 6.6. ∼150 lines.

Step 10: Numerical readout.
Prove Proposition 7.1 and Corollary 7.2. ∼80 lines.

Step 11: Split-complex readout and $\tilde{\chi}$.
Define $\mathbb{D}$, $\mathbf{e}_+$, $\mathbf{e}_-$ (Definition 7.3); define χ, $\tilde{\chi}$ (Definitions 7.4, 7.7); prove Lemma 7.6 (via Book II Chapter 47 [4] idempotent-completeness), Proposition 7.8 with *additive target* $(\mathbb{D}, +)$ and both trace identities, and Corollary 7.14 (finite-stage trace identity with ε_n vanishing at primorial depths, plus ω-limit passage via continuity of Tr_+ on $\mathbb{D} \otimes \mathbb{R}_\tau$). ∼180 lines.

Step 12: Saturation.
Prove Conjecture 8.2. This step depends on Book VII ch. 48 infrastructure (self-enrichment tower) and is provisional until that chapter is formalised in TauLib. ∼120 lines (skeleton), ∼400 lines (full).

Line-count estimate (v2).. Steps 1–11: $\approx 100+80+200+150+100+220+120+60+60+80+150+80+180 = 1580$ Lean lines for the core derivation. Step 12 adds another ∼120–400 lines depending on whether the Book VII self-enrichment infrastructure is co-formalised. Estimated at three formalisation sprints (10–12 weeks) assuming the σ-equivariant holomorphic transformer API and the polarity lattice $\Lambda[n]$ are newly constructed.

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