

# The Natarajan Dichotomy as One D-Sector Readout in Two Regimes

*A  $\tau$ -Categorical Pre-Registration against the Natarajan, Chiang & Dutra (2026) Multiscale Cluster-Lensing CDM Crisis*

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## 1. WHY THIS PAPER

The Natarajan et al. (2026) CDM-crisis paper [1] differs from prior small-scale tensions in two respects that matter for any non-particle response.

**Observational robustness..** The  $5\sigma$ – $40\sigma$  inner projected substructure excess at  $R/R_{200} \lesssim 0.2$  survives the authors'  $10^6$ -iteration bootstrap injection of force-resolution-disrupted inner subhalos (their Appendix B), and the order-of-magnitude tension in GGSL probability  $P_{\text{GGSL}}$  persists under varied baryonic-feedback prescriptions (their §9.3, citing [2, 3, 4, 5]). This is not a marginal anomaly within a noisy substructure inventory; it is a stress test of  $\Lambda$ CDM at the dense-cluster intermediate-redshift regime, performed under maximal  $\Lambda$ CDM-favourable conditions (TNG-Cluster, baryonically self-consistent, force-disruption-aware).

**Structural distinctness..** Unlike the missing-satellites or cusp–core problems, the Natarajan tension demands that the *same* subhalo host two mutually exclusive microphysical signatures: *collisional-like* inner density slopes ( $\gamma \gtrsim 2.5$ – $3$  at  $r \lesssim 0.01 R_{200}$ , core-collapsed SIDM regime) and *collisionless-like* outer tidal extents ( $r_t$  at  $r \sim 10$ – $100$  kpc, consistent with NFW dynamical friction). No single self-interaction cross-section  $\sigma/m$  satisfies both [1] §10.

**Open door to non-particle alternatives..** The authors [1] §10 explicitly foreclose the conventional SIDM resolution — a sophisticated structural commitment we gratefully acknowledge — in concert with the Chiang+2026b [6] companion analysis, and nominate either a hybrid CDM+SIDM model or “*entirely new classes of DM theories*” as candidate resolutions. This opens the response landscape to non-particle frameworks — where the present Panta Rhei Programme is

already positioned via the Category- $\tau$  no-particle-DM commitment of [7] (cluster-scale lensing without particle dark matter, V.T213 universality at  $M_{\text{eff}}/M_p \leq 6.65$ ). A Category- $\tau$  note can therefore engage substantively, with cluster-scale precedent already in the public corpus, and without competing inside the saturating SIDM-core-collapse debate.

What follows is a categorical pre-registration in the KSZ [7] and LRD-v3.3 [8] style: three  $\tau$ -distinctive positive readings, one numerical anchor, one headline falsifier, three pre-registered gaps, and one structurally honest tension re-framed as falsifier.

## 2. THE $\tau$ -POSITION IN ONE LINE

**No particle DM..** The Panta Rhei Programme commits to a capacity-gradient framework in which dark-matter phenomenology is a projection artefact of the categorical  $D$ -sector readout, not a particle species. The framing was pre-registered against the Gallardo et al. pairwise-kSZ force-law constraint [7], where the Newtonian-limit theorem V.T78 fixes the large-scale exponent at  $n = 2$  (Newtonian), in tension with MOND  $n = 1$ .

**KSZ cluster-scale precedent (with substructure caveat)..** [7] commits the framework to cluster lensing phenomenology via the boundary-holonomy mass closure  $M_{\text{eff}}/M_p \leq 6.65$  (V.D272 + V.T212 [READOUT]), audited against a five-cluster universality catalogue (V.T213). That catalogue is *four merging plus one relaxed* (Bullet, A2744, El Gordo, MACS J0025; plus A1689), at total-mass per cluster, with  $\pm 20$ – $50\%$  precision on integrated cluster mass only (not substructure-resolved). The present three Natarajan clusters are characterised as relaxed in [1], with MACS J0416's dynamical state contested in the broader HFF literature, and the observable is substructure-resolved at kpc scale. The KSZ V.T213 universality is therefore inherited here only as a calibration prerequisite on integrated cluster mass, not as a substructure-resolved theorem.

**LRD-v3.3 format precedent..** The pre-registration scaffold follows [8], which attaches eight observational signatures to the heavy-seed black-hole birth condition at  $z \approx 8-15$ . Signature 3 of that note ( $M_{\text{eff}}/M_p$  aperture-dependent saturation at LRD scales) shares its anchor (V.D272 + V.T213) with the present §3 below; to avoid silent double-booking, the LRD coupling is disclosed in Ni6.3 of the present note (see §5).

**Status discipline..** Every  $\tau$ -statement on Natarajan observables in this note carries [READOUT] (consistency reading or interpretation, not external-physics theorem) or [EVS] (external-validation seam, falsifier). No [FORMAL] or [HINGE] claim is made on the  $\gamma$  value, the SHMF amplitude, the  $r_t$  value, or the  $5\sigma-40\sigma$  amplitude. The V.T203 conjectural status (cluster capacity mass discrepancy,  $D_\tau \sim 2-4$  vs.  $D_{\text{obs}} \sim 5-7$  [7] L8II-8I4) is carried forward verbatim; this note does not narrow that gap, only structurally locates it (see §6).

### 3. TWO REGIMES, TWO MECHANISMS

The Natarajan dichotomy presents at face value as a microphysical paradox: the same dark-matter sector must be *collisional* in inner cores and *collisionless* in outskirts. The categorical reading is different. There is one  $D$ -sector readout; it produces two regime-separated nonlinearities at the cluster-substructure scale, organized by a single length scale  $\ell_{\text{cl,sub}}$  that depends on subhalo mass, not on a particle parameter. Inside  $\ell_{\text{cl,sub}}$  the readout is dominated by inner promotion of the boundary-holonomy mass  $M_\partial(R)$  in the host capacity well. Outside  $\ell_{\text{cl,sub}}$  the capacity-screening factor relaxes to unity and the Newtonian-limit theorem governs.

**One readout, not two..** The categorical position is that the dichotomy is an *artefact* of asking the question in particle-DM language. In Category  $\tau$  there is no second component, no density-dependent cross-section, no hybrid; the inner and outer phenomenologies are two regime-projections of one underlying readout (V.D121 capacity skeleton + V.D272 boundary-holonomy mass closure + V.D261 cluster screening). The three subsections below make this concrete.

#### 3.1 Inner regime: $M_\partial(R)$ inner-promotion

**Mechanism..** At cluster radii  $R/R_{200} \lesssim 0.2$ , the host capacity gradient is maximal and the boundary-holonomy mass  $M_\partial(R)$  is enhanced relative to its outskirts value (V.D272 [READOUT]). The capacity-skeleton population of sub-threshold peaks is thereby *preferentially promoted* above the spectroscopic luminosity threshold inside this projected radius. The result is an excess in the projected member-galaxy count at  $R/R_{200} \lesssim 0.2$  relative to a  $\Lambda$ CDM expectation that carries no  $M_\partial$  at all (the TNG-Cluster suite, [9], solves only the standard  $\Lambda$ CDM Einstein-Boltzmann system; the boundary-holonomy contribution is structurally absent).

**Why this is sign-correct..**  $\Lambda$ CDM substructure suffers numerical disruption inside  $\sim 0.2R_{200}$  even after Hopkins-style force-resolution-free reanalysis (see [1] App. B and [10, 11]);

the resulting simulated radial distribution carries a residual inner *dearth*, not an inner excess. The Category- $\tau$  direction is opposite:  $M_\partial(R)$  inner-promotion concentrates the readout inward. The amplitude is not derivable from existing primitives ( $M_\partial(R)$  functional form deferred to a future Book V cluster-interior module); the *direction* is structurally guaranteed by V.D272 boundary-holonomy enhancement at the cluster nodal-capacity maximum.

**V.T203 conservatism..** The V.T203 conjectural status [7] L8II-8I4 (cluster capacity mass discrepancy,  $D_\tau \sim 2-4$  vs.  $D_{\text{obs}} \sim 5-7$ ) governs the *integrated* cluster-scale mass closure, not the projected radial distribution of peaks. The present §3.1 therefore neither narrows nor inherits V.T203, but *relocates* its empirical signature: the mass discrepancy can in principle be partially absorbed by the boundary-holonomy enhancement near the cluster centre, which has not been previously quantified.

#### 3.2 Outer regime: Newtonian-limit on tidally-pruned capacity wells

**Free consistency check..** At radii  $r \sim 10-100$  kpc, the cluster-scale screening factor is

$$1 + \frac{r}{\ell_\tau} = 1 + \frac{r}{c/(H_0\sqrt{1-\nu_\tau})} \approx 1 + (1.6-16) \times 10^{-5}$$

(using the cosmological identity  $\ell_\tau = c/(H_0\sqrt{1-\nu_\tau}) \approx 5.48$  Gpc [7], V.D261).<sup>2</sup> The Roche-like tidal expression for subhalo truncation (Book V ch.36 [READOUT]) therefore reduces to the standard Newtonian form to one part in  $\sim 10^5$ , and the predicted tidal-truncation-radius distribution is collisionless-CDM-like *by construction*. Natarajan's Figure 3 (the SIDM  $r_t$  distribution sits nearly an order of magnitude below the CDM-like observed band; the specific median  $r_t^{\text{SIDM}}/r_t^{\text{CDM}} \approx 0.73$  at the  $5\sigma$  rejection level is reported in Chiang+2026a [11]) is therefore matched without invoking dark-sector self-interactions; this is a consistency check on the dichotomy's *outer* side, not a derivation of the specific  $r_t$  value.

**What this does and does not commit..** This subsection commits the framework only to *statistical consistency with CDM at outer subhalo radii*, which is identical to the collisionless-CDM expectation that Natarajan et al. find satisfied. It does not commit a value of  $r_t$  for any specific subhalo, nor an amplitude of the  $r_t$  distribution width. Those follow from the host density profile and orbital-pericentre distribution, which are external inputs to the present [READOUT].

<sup>2</sup>The V.D261 entry in the corpus historically carried a unit-label slip (Mpc vs. Gpc) documented in [7] §6; the present note adopts the corrected Gpc-scale identity consistently. The dressed value incorporating the V.T207  $\sqrt{1-\nu_\tau}$  pre-factor lies in the range 5.48-13.0 Gpc; the figure quoted here is the bare value appropriate for the screening-factor derivation. The  $K_0(r/\ell_\tau)$  near-field profile of V.T85 is the Yukawa-Stokes signature of a massive propagator ( $\eta_\tau = 0$ , mean-field universality class), not a marginal-RG inheritance — a distinction sharpened in the Phase 13 Path C' synthesis and disjoint from cousin-paper marginality results.

### 3.3 Transition scale $\ell_{\text{cl,sub}}$

**Closed-form prediction..** The transition between the inner  $M_\partial$ -dominated regime (§3.1) and the outer Newtonian-limit regime (§3.2) is governed by the *classical-validity length applied to the member subhalo's own mass*:

$$\ell_{\text{cl,sub}}(M_{\text{sub}}) = \sqrt{\frac{GM_{\text{sub}}}{a_0}}, \quad a_0 = \frac{cH_0\iota_\tau}{2} \text{ (V.D232)}, \quad (i)$$

where  $\iota_\tau$  is the master  $\tau$ -constant (Book V V.D232 in the capacity-gradient framework [7]). The host-cluster  $\ell_\tau$  (Gpc scale; see above footnote) and the host-cluster classical-validity scale  $\ell_{\text{cl}}(M_{200}) \approx 1.2 \text{ Mpc} \approx 0.45 R_{200}$  are both *outside* the inner transition; the empirically relevant length in the cluster-substructure context is the *member* scale  $\ell_{\text{cl,sub}}(M_{\text{sub}})$ , not the host scale.

**Numerical anchor..** For  $M_{\text{sub}} \in [10^{10}, 10^{11}] M_\odot$  (the dominant mass range of the Natarajan spectroscopically-confirmed member sample as tabulated in [1] §§2–3 and Fig. 1; the underlying SHMF extends to  $\sim 10^{13} M_\odot$ , but the spectroscopic and dynamical weight in the GGSL-relevant cluster-member population sits at the lower end),

$$\ell_{\text{cl,sub}} \in [3.4, 10.8] \text{ kpc},$$

in order-of-unity agreement with the empirical break at  $r \sim 10 \text{ kpc}$  across all three MACS clusters (MACS J0416, J1206, J1149). The match is *categorical* (the existence and location of a transition scale, not its precise value): Category  $\tau$  predicts a transition,  $\Lambda\text{CDM}$  and MOND predict none, and the observed break sits in the predicted band. The host concentration ( $c, r_s$ ) dependence of  $\ell_{\text{cl,sub}}$  is deferred to a closed-form derivation via the full V.D259 nonlinear capacity equation (a future Book V cluster-interior module); the present order-of-unity match is robust to the host- $c$  variation across  $c \in [3.5, 8]$  at the level of  $\pm 0.3$  dex.

**Distinguishability of the transition scale..** The closed-form (i) is structurally a MOND-proxy at the *subhalo* (member-galaxy) scale, where MOND traditionally works. This is *not* the host-cluster scale where MOND fails the missing-mass test (Famaey, Pizzuti & Salas [12], Fig. 4 of which shows residual missing mass in CLASH-sample relaxed clusters with central density ratio  $\sim 10\times$  the gas density at inner Mpc). The Category- $\tau$  inheritance of MOND-proxy is therefore regime-bounded:

- At subhalo  $M_{\text{sub}} \sim 10^{10}–10^{11} M_\odot$ :  $a_0$ -scale validity holds (V.P68 + V.R175 [7]);  $\tau$  inherits the MOND-proxy transition radius.
- At host-cluster  $M_{200} \sim 10^{15} M_\odot$ : the validity regime is exited; the cluster-scale failure mode of MOND [12] is recovered into V.T203's conjectural slot, not silently absorbed.

SIDM core-collapse predicts  $r_{\text{trans}}^{\text{SIDM}} \propto (\sigma/m)^\alpha$  with no scaling against  $M_{\text{sub}}$  at fixed cross-section; Category  $\tau$  predicts

$r_{\text{trans}} \propto M_{\text{sub}}^{1/2}$ . The *host-mass independence* of  $r_{\text{trans}}$  at fixed subhalo mass function is the headline pre-registered falsifier (N16.3 of §5).

## 4. CONSISTENCY CHECKS PASSED

Three orthogonal Natarajan diagnostics admit Category- $\tau$  consistency readings at the [READOUT] level.

**§4.1 Outer  $r_t$  at 10–100 kpc.** The screening factor of §3.2 reduces the Book V ch.36 Roche expression to the standard Newtonian form at one part in  $\sim 10^5$ . The Natarajan Figure 3 SIDM-vs-CDM distribution (with the Chiang+2026a companion's  $r_t^{\text{SIDM}}/r_t^{\text{CDM}} \approx 0.73$  median at the  $5\sigma$  rejection level [11]) is matched by construction. A consistency check, not a derivation of any specific  $r_t$ .

**§4.2 Inner member-number excess at  $R/R_{200} \lesssim 0.2$ .**

The boundary-holonomy mass  $M_\partial(R)$  inner-promotion mechanism (§3.1) produces a sign-correct concentration at small projected radii. The Natarajan Figure 2  $5\sigma–40\sigma$  excess is matched in direction; amplitude is deferred ( $G_3$  in §6). The Fr-wave  $M_\partial(R)$  structural-closure analysis (Phase 12, V.D352 + V.T348 [1] cross-reference) predicts a  $\tau$ -side inner-excess band consistent with the *inner half* of the Natarajan band ( $\sim 5\sigma–20\sigma$ ), with the high-tail residual ( $\gtrsim 30\sigma$ ) plausibly absorbing into the slope-axis contribution that the V.D272 inner-promotion mechanism cannot supply. TNG-Cluster [9] cannot reproduce this structurally because the underlying Einstein–Boltzmann system carries no  $M_\partial(R)$  contribution; the  $5\sigma–40\sigma$  residual survives the  $10^6$ -iteration Hopkins-style bootstrap correction in Natarajan Appendix B [1, 10], so the force-resolution-loss escape hatch is already closed by the source analysis before the  $\tau$ -mechanism is invoked. Count-axis and slope-axis contributions are partially degenerate at present single-cluster precision; the count-vs-slope joint inference roadmap is §6 G1.

**§4.3 Transition scale  $r_{\text{trans}} \approx 3.4–10.8 \text{ kpc}$ .** The closed-form  $\ell_{\text{cl,sub}}$  of (i) matches the empirical break at  $r \sim 10 \text{ kpc}$  across the three MACS clusters to order unity, for the  $10^{10}–10^{11} M_\odot$  subhalo mass range that dominates the spectroscopically confirmed sample. The Natarajan §7 inner-density transition and §8 outer truncation-radius transition share this single length scale.

## 5. PRE-REGISTERED FALSIFIERS (N16 SURFACE)

We register the following falsification surface (N16.1–N16.11) as the cluster-substructure  $D$ -sector dichotomy entry in the Panta Rhei falsification ledger, continuing the N9–N15 sequence from [7] (N12–N14) and [8] (N15). Rows N16.1–N16.8 were registered in the original research wave (R5 falsification-surface refinement); N16.9 (joint  $\iota_\tau$  universality) and N16.10 ( $M_\partial(R)$  radial shape) were added at Phase 9

(Workstream C); and N16.11 (concentration-dependent saturation radius  $R_*(c) = 1.25 R_{200}/c^2$ ) was added at Phase 12 from the V.D352 + V.T348 Chern–Weil partial-domain holonomy structural closure. Each row is [EVS] and is testable with named near-term surveys (Euclid, Rubin LSST, JWST cluster-lensing follow-up, FRBs, SKA, Roman, CMB-S4). The headline row is N16.3; the consistency floor is N16.1.

**Headline: N16.3, host-mass independence of  $r_{\text{trans}}$ .** The discriminating prediction is the cluster-mass dependence of the transition radius. Category  $\tau$  predicts  $r_{\text{trans}} \propto M_{\text{sub}}^{1/2}$  with no explicit host-mass scaling at fixed subhalo mass function; SIDM predicts  $r_{\text{trans}} \propto (\sigma/m)^\alpha$  (subhalo-mass-independent at fixed cross-section); CDM predicts no transition at all; MOND predicts the wrong scale at the host mass. The test requires a sample of  $\gtrsim 30$  massive lensing clusters with strong+weak lensing and substructure-resolved inner profiles (consistent with the Meneghetti +2020 / Despali +2017 survey-forecast standard [2]; a tighter band-comparison  $\chi^2$  formulation may reduce this to  $\sim 20$  in a future power-calculation pass, but the conservative figure stands here); N16.9 adds  $\sim 50$  SLACS-class galaxy-scale lenses plus  $\sim 30$  LRD lenses on top, with the shared  $\nu_\tau$  parameter delivering a tighter joint constraint than any single survey delivers alone. The Euclid wide survey plus Rubin LSST Year-2 [13, 14] deliver the first sub-sample by 2027–2029; the full headline test also requires JWST + Roman kpc-resolved  $\gamma(r)$  stacking on the 2028–2030 horizon, not Euclid+LSST alone.

**Consistency floor: N16.1, outer  $r_t$ .** The Roche-like tidal extents at  $r \sim 10$ –100 kpc are matched by construction in Category  $\tau$  (§3.2); the Natarajan inference here is therefore *consistent*, not predictive. This row survives even if all other R-wave outcomes degrade to negative results.

**V.D272 / V.T213 coupling disclosure and  $\nu_\tau$  joint-prediction lock.** N16.3 and N16.6 share their boundary-holonomy mass anchor (V.D272) with [8] Signature 3, which attaches the same anchor to aperture-dependent saturation at LRD scales. The coupling is *quantitative*: both the LRD signature-3 6.65 saturation ceiling and the cluster-inner  $M_\partial(R)$  amplitude in N16.2/N16.4 depend on the same  $\tau$ -master-constant pre-factor  $\nu_\tau = 2/(\pi + e) \approx 0.341$  via V.D272. The two predictions are therefore *not independently falsifiable*: a single  $\nu_\tau$  parameter generates two empirical signatures on different observables, giving a positive consistency check at the cost of joint exposure (N16.9). Any null result on the LRD-side SLACS-saturation prediction propagates immediately to a tension on the present  $M_\partial(R)$ -based inner-promotion; conversely, any aperture-independent  $M_\partial(R)$  measurement would falsify both predictions jointly and force a structural revision of V.D272. This coupling is disclosed here per the R4 conservatism ledger and the F5 falsification-surface refinement.

**Row-independence audit.** The eleven N16 rows span approximately five independent physical axes, not eleven. The intra-N16 couplings are: N16.3  $\leftrightarrow$  N16.4 (both anchored on  $r_{\text{trans}} = \ell_{\text{cl,sub}}$ ; a null on N16.4 cascades to N16.3); N16.2  $\leftrightarrow$  N16.7  $\leftrightarrow$  N16.10 (all anchored on  $M_\partial(R)$  inner-promotion; a null on N16.2 propagates to N16.7 and N16.10); N16.11  $\leftrightarrow$  N16.10 (both probe  $M_\partial(R)$  radial shape). The shared  $\nu_\tau$  pre-factor across N16.2, N16.4, N16.6, N16.9 (V.D272 + LRD-S3 joint lock above) is the cross-program coupling. Net: the eleven rows resolve to roughly *five independent axes* — transition scale,  $M_\partial(R)$  inner-promotion, SHMF slope/amplitude, V.T203 cluster mass discrepancy, and FRB scintillation. The shared  $\nu_\tau$  joint exposure is, equivalently, a *joint parameter constraint*: a future  $\sim 50$ -SLACS +  $\sim 30$ -LRD +  $\sim 30$ -cluster joint fit on  $\nu_\tau$  is a tighter single-parameter constraint than any single survey delivers.

**Timeline tiers.** First decisive data per row: 2026–2028 (JWST cluster-lensing follow-up of Frontier-Fields and CLASH targets via the BUFFALO, MAGNIF, CANUCS, and GLASS-JWST programs); 2027–2029 (Euclid wide survey plus Rubin LSST Year-2 first sub-sample); 2028–2030 (full Euclid + LSST + JWST + Roman + FRB cluster sightlines); 2035+ (SKA, Roman wide, CMB-S4).

## 6. HONEST GAPS AND THE STRUCTURALLY HONEST TENSION

The wave that produced this note (research-wave/02-R1...R5) explicitly closed two of the three binding gaps from the prior red-team triage at the partial level (direction-and-mechanism), and produced one numerical anchor ((1)). It also returned one principled negative finding that we now reframe as a falsifier.

**G1 — inner  $\gamma$  value (R2 NEGATIVE; reframed).** The natural candidate mechanism for steep inner cusps in Category  $\tau$  was the *saturation family* of Book V (V.T36, V.D57, V.P16, V.T117). On re-inspection the family is *structurally anti-cusp*: V.P16 explicitly replaces every configuration that would otherwise produce a general-relativistic density singularity with bounded density. The Schwarzschild-radius scale  $R_S(M_{\text{sub}} \sim 10^{11} M_\odot) \sim 10^{-8}$  kpc is 4–7 decades below Natarajan’s target  $r \lesssim 0.01 R_{200} \approx 1$ –10 kpc, so the saturation-replaces-singularity construction is geometrically inert at the relevant scale. The V.T117 saturation radius is a *cosmological motif-saturation* length ( $\sim 100$  Mpc scale), not a kpc-scale halo-core radius. The V.D305 fiber-holonomy primitive cited by triage agent A1 as a candidate cross-context primitive turns out to be the BBN  ${}^7\text{Be}$  radiative-capture correction (module BookV . Cosmology . BBNuclearNetwork), not a saturation primitive; the actual Wave-R7 LRD-1 heavy-seed derivation uses V.D-LRD-1d, V.T88, V.T108, V.T110 plus  $\Pi_{\text{coh}}$  ([8]), all rotational/threshold mechanisms with no density-slope crossover. The categorical conclusion is that *Category  $\tau$  predicts flatter, not steeper, inner density slopes at  $r \lesssim$  kpc in*

**Table 1.** Ni6 falsification surface — cluster-substructure  $D$ -sector dichotomy entry. Rows Ni6.1–Ni6.8 originate from the R5 falsification-surface refinement; Ni6.9–Ni6.10 were added at Phase 9 (Workstream C); Ni6.11 was added at Phase 12 from the V.D352 + V.T348 Chern–Weil partial-domain holonomy structural closure. Every row is [EV5].

ID	Prediction	$\tau$	CDM	SIDM core-collapse	MOND
Ni6.1	Outer $r_t$ at 10–100 kpc NFW-Roche-like	match (§4.1)	match	$\sim 10\times$ smaller	ambiguous
Ni6.2	Inner $R/R_{200} \lesssim 0.2$ <i>number</i> excess	match (§4.2); 5 $\sigma$ –20 $\sigma$ band	5–7 $\times$ , dearth (TNG-C)	dearth	no prediction
Ni6.3	<b>Host-mass independence of <math>r_{\text{trans}}</math> at fixed SHMF</b>	$r_{\text{trans}} \propto M_{\text{sub}}^{1/2}$ ( <i>headline</i> )	no $r_{\text{trans}}$	$\propto (\sigma/m)^\alpha$	wrong scale
Ni6.4	Transition $r_{\text{trans}} \approx \ell_{\text{cl,sub}} \approx$ 3.4–10.8 kpc	match (§4.3, (1))	no $r_{\text{trans}}$	$M_{\text{sub}}$ -independent	host-scale, not subhalo-scale
Ni6.5	SHMF slope $dn/dM_{\text{sub}} \propto M_{\text{sub}}^{-1.9}$	slope $\sim -2$ (capacity peaks); $A_\tau/A_{\text{TNG-C}} \in [0.5, 1.5]$ per V.T13 floor	match	suppressed at low $M_{\text{sub}}$	no prediction
Ni6.6	Cluster-scale missing-mass profile within central 1 Mpc	V.T203 conjectural slot [7]	full match	residual	$\sim 10\times$ gas, inner core [12]
Ni6.7	FRB scintillation: cluster sightlines through $M_\partial$ peaks	dispersion-measure bumps where $M_\partial(R)$ peaks	smooth	smooth	no prediction
Ni6.8	GGSL probability scaling with $M_{\text{sub}}$	member-density dominated	low (dearth)	slope-dominated (cusp)	no prediction
Ni6.9	<b>Joint <math>\nu_\tau</math> universality across LRD + cluster samples</b>	shared pre-factor $\nu_\tau = 2/(\pi + e) \approx 0.341$ at both LRD aperture-saturation and cluster $M_\partial(R)$	no relation	no relation	no relation
Ni6.10	Radial-shape discriminator on $M_\partial(R)$	shape between $1/R$ and $1/R^3$ inside $R_{200}$	no $M_\partial$	no $M_\partial$	no $M_\partial$
Ni6.11	Concentration-dependent saturation radius $R_*(c) = 1.25 R_{200}/c^2$	$R_*/R_{200} \in [0.02, 0.10]$ for $c \in [3.5, 8]$ ; derived from NFW partial-domain Chern– Weil	no $M_\partial$	no $M_\partial$	no $M_\partial$

*cluster subhalos*, on the *opposite* side of Natarajan’s parametric  $\gamma \gtrsim 2.5$  dPIE inference (whose direct empirical anchor is Dutra et al. 2025 [15],  $\gamma \sim 2.9$  across three relaxed MACS clusters) from SIDM core-collapse. **Reframe as a conditional discriminating observation (not a dissolution of G1).** The GGSL probability  $P_{\text{GGSL}}$  is sensitive to the inner mass distribution *integrated* along the line of sight; it carries contributions from *two distinct axes* that are partially separable in joint GGSL+kinematics inference (the prototype technique is Granata et al. 2022 [16]’s MUSE+HST joint strong-lensing + member-velocity-dispersion modelling on MACS J0416, with current state-of-the-art HFF/CLASH lens models in Bergamini et al. 2023 [17]):

**Count axis.** The number and spatial concentration of member-galaxy subhalos providing GGSL targets. Category  $\tau$  addresses this axis sign-correctly via the  $M_\partial(R)$  inner-promotion mechanism (§3.1).

**Slope axis.** The per-subhalo inner density profile  $\gamma$ . SIDM core-collapse addresses this axis by steepening; Category  $\tau$  predicts *flatter*, not steeper, slopes via V.P16. Importantly, Tokayer et al. (2024) [18] already exhausted the *count-axis-only* approach via intra-cluster baryonic mass rearrangement and found it insufficient to match the observed  $\times 3.5$ – $\times 8$   $P_{\text{GGSL}}$  boost in Natarajan §7. Thus

the  $\tau$ -readout cannot dissolve G1 as a category error; the slope axis is empirically required to be *somewhere*, and the conditional question is where.

The discriminating test, as a conditional:

[READOUT] *At fixed inferred  $P_{\text{GGSL}}$ , does the GGSL+inner-mass-content joint inference attribute the signal predominantly to count axis ( $\tau$  via  $M_\partial(R)$  inner-promotion) or to slope axis (SIDM core-collapse with  $\gamma \gtrsim 2.5$ )? If predominantly count axis, Category  $\tau$  matches without invoking a steep inner slope; if predominantly slope axis,  $\tau$  requires the deferred metric-sector progenitor of §6 (V.P144 + V.D123 trace-projection; program-level open item) before answering. Galaxy-scale corroboration of core-collapsed compact substructure is independently emerging outside the cluster regime (Vegetti et al. 2026 on JVAS B1938+666 via VLBI [19]; Powell et al. 2025 via ALMA [20]), which Natarajan et al. §10 cite as cross-context evidence; the present note remains cluster-scale-only. Future JWST cluster-lensing + kinematic IFU follow-up will discriminate (BUFFALO, MAGNIF, CANUCS, GLASS-JWST programs are the near-term*

*anchor*;  $\sim$  2028–2030 horizon).

**G2 — SHMF amplitude (R1 PARTIAL)..** The slope  $\sim -2$  of the cluster member-galaxy capacity-peak abundance follows from peak statistics of the CMB/BAO-constrained Gaussian random field over the underlying capacity field (R1 derivation, [READOUT]); the recovered slope agrees with the Natarajan-observed  $dn/dM_{\text{sub}} \propto M_{\text{sub}}^{-1.9}$  to within slope uncertainty. The *amplitude* of the SHMF is not derivable from existing primitives. Flagged for future Book V cluster-interior scaffolding.

**G3 — inner-excess amplitude (R1 PARTIAL)..** The sign-correct inner concentration of member counts at  $R/R_{200} \lesssim 0.2$  via  $M_{\partial}(R)$  inner-promotion (§3.1) is structurally guaranteed; the  $5\sigma$ – $40\sigma$  *amplitude* is not derivable from existing primitives. Flagged for future Book V scaffolding.

**Pre-registration discipline..** The three gaps above are *not* promised against existing primitives. A future Book V module — tentatively “Cluster Member Distribution from the Capacity Skeleton with Boundary-Holonomy Inner Promotion” — would close G2 and G3 simultaneously; closing G1 requires a genuinely new *boundary-holonomy radial-steepening* mechanism distinct from both V.D272 inner-promotion (which is a number-density mechanism) and the saturation family (which is structurally anti-cusp). Such mechanism, if it exists, would lift this note to the deferred F3 form factor. As of the present authoring it does not exist in the published  $\tau$ -corpus, and the structurally honest position is to commit to G1’s reframe-as-falsifier rather than to scaffold a new primitive on demand.

## 7. DISTINGUISHABILITY STATEMENT

Category  $\tau$  is structurally distinct from every active alternative on the Natarajan landscape:

**CDM (collisionless).** Same outer  $r_t$  prediction by construction (§3.2); different inner mechanism — CDM has no  $M_{\partial}(R)$  and predicts the observed *dearth*, not excess; no transition scale.

**Constant- $\sigma$  SIDM.** Different mechanism — saturation (anti-cusp) vs. collisional steepening; opposite  $\gamma$ -direction (§6 G1); transition scale parameter dependence:  $(\sigma/m)$  vs.  $M_{\text{sub}}^{1/2}$  (N16.3).

**Velocity-dependent SIDM.** Same as constant- $\sigma$  on the  $M_{\text{sub}}$ -vs- $\sigma/m$  scaling discriminator *up to a residual*  $v_{\text{disp}}(M_{\text{sub}})$  *running* via the Rutherford-like cross-section of Yang & Yu 2022 [21] adopted in Nadler et al. 2025; this running is much weaker than  $M_{\text{sub}}^{1/2}$  and carries the *opposite sign* in the mildly-collisional regime (smaller subhalos have lower  $v_{\text{disp}}$ , hence larger effective  $\sigma/m$ , hence *earlier* core-collapse and smaller  $r_{\text{trans}}$ ). The mildly-collisional long-lasting core-phase mechanism of

Errani et al. 2023 [22] is what closes Natarajan §8’s vel-dep escape hatch. The N16.3 discriminator therefore survives against vel-dep SIDM, but the foil is not identical to constant- $\sigma$ . Same as constant- $\sigma$  on the cusp/anti-cusp  $\gamma$ -direction discriminator. Outer  $r_t$ -truncation tighter than CDM in vel-dep cross-section models [23]; Category  $\tau$  predicts CDM-like by construction.

**MOND.** V.T78 force-law exponent  $n = 2$  vs. MOND  $n = 1$ ;  $a_0$ -scale validity regime-bounded (§3.3): inherited at subhalo  $M_{\text{sub}} \sim 10^{10}$ – $10^{11} M_{\odot}$ , exited at host  $M_{200} \sim 10^{15} M_{\odot}$  where MOND fails the cluster-mass test [12]. Category  $\tau$  does *not* invoke a universal  $a_0$  across all scales; the  $\ell_{\text{cl,sub}}$  length *is* a  $\sqrt{GM_{\text{sub}}/a_0}$  expression but with  $a_0$  a derived constant from the cosmological capacity-gradient framework (V.D232, V.P68, V.R175 [READOUT]), not a postulate.

**Hybrid CDM+SIDM.** Ontologically distinct — one  $D$ -sector, no second component, no density-dependent cross-section (§3). Observationally distinct on *host-concentration* dependence of  $r_{\text{trans}}$ : hybrid CDM+SIDM still carries a density-dependent transition (slope axis activates inside  $\rho_{\text{crit}}$ ), giving a  $r_{\text{trans}}$  that depends on host concentration  $c$  at fixed  $M_{\text{sub}}$ ; Category  $\tau$ ’s  $r_{\text{trans}} \propto M_{\text{sub}}^{1/2}$  scaling carries no explicit host- $c$  dependence. The two are discriminable in a  $\sim 30$ -cluster sample across the  $c \in [3.5, 8]$  range of relaxed CLASH/HFF cluster lenses.

**Verlinde / MOG.** Different mechanism class — Category  $\tau$  inherits the Newtonian  $n = 2$  force law from V.T78 [7], not entropic gravity or Yukawa-modified gravity; no parametric free parameter to tune against cluster lensing.

The minimum testable distinguishability set within the N16.I–N16.II surface (§5) is N16.3 (Category  $\tau$  uniquely predicts host-mass independence of  $r_{\text{trans}}$  at fixed SHMF) plus N16.2 ( $M_{\partial}(R)$  inner-promotion produces excess; all four alternatives produce dearth or no prediction). Combined, these two rows form a minimum-cluster-sample-size discriminator of  $\sim 30$  massive clusters at strong+weak lensing precision, which Euclid+LSST will deliver by  $\sim 2029$ .

## 8. STATUS DISCIPLINE

Every  $\tau$ -claim in this note is tagged [READOUT] or [EVS]; no [FORMAL] or [HINGE] claim is made on the Natarajan empirical observables. The section-to-tag mapping is given in Table 2.

V.T203 (Cluster Capacity Mass Discrepancy) inherits its skeleton/conjectural status from [7] L8II–814 verbatim; this note neither narrows nor extends V.T203, only structurally relocates its empirical signature (§3.1 *V.T203 conservation* paragraph). The V.T213 five-cluster universality is inherited as a calibration prerequisite on integrated cluster mass, not as a substructure-resolved theorem (§2 *KSZ cluster-scale precedent* paragraph).

**Table 2.** Status-tag mapping ([READOUT] = consistency reading / interpretation; [EVS] = external-validation seam / falsifier).

Section	Claims	Tag
§3.1 ( $M_{\partial}(R)$ inner-promotion)	sign-and-mechanism	read- OUT
§3.2 (outer $r_t$ via Newtonian limit)	consistency check	[READ- OUT]
§3.3 (transition $\ell_{cl,sub}$ )	numerical reading	[READ- OUT]
§4 (three consistency checks)	all §4.x	[READ- OUT]
§5 (N16.1–N16.11)	future-observation	[EVS]
§6 ( $G_1$ reframed; $G_2$ , $G_3$ flagged)	pre-registered gaps + reframed falsifier	[EVS]
§7 (foil paragraphs)	positioning	[READ- OUT]

## 9. WHAT THIS NOTE DOES NOT CLAIM

1. It does *not* claim Category  $\tau$  derives the  $\gamma \gtrsim 2.5$  inner density slope, the SHMF amplitude, the  $5\sigma$ – $40\sigma$  inner-excess amplitude, or the specific  $r_t$  distribution width inferred by Natarajan et al. from any existing  $\tau$ -corpus primitive. Each is flagged as a pre-registered gap (§6).
2. It does *not* claim that  $\tau$ -physics is Lean-verified as external physics. The cluster-substructure context is outside the scope of the Lean-formalized  $\tau$ -internal-mathematics envelope (see the dossier [24] App. C).
3. It does *not* claim an externally-settled resolution of the small-scale CDM tensions or of any Clay Millennium problem related to gravitational lensing or cluster physics.
4. It does *not* predict the IllustrisTNG-Cluster simulation [9] as a  $\tau$ -readout; the simulations are external  $\Lambda$ CDM-expectation comparators, not  $\tau$ -derivations.
5. This note is a privately-deposited arXiv response in the Panta Rhei research-note format and is not a substitute for direct dialogue, which the authors actively welcome and to which the present pre-registration is offered as an opening contribution.
6. It does *not* supersede the conjectural status of V.T203 in [7]; the V.T203 cluster mass discrepancy remains an open conjectural item, structurally relocated but not narrowed by the present note (§3.1).

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here. Engagement, criticism, and falsification of any of the eleven pre-registered N16 rows would be welcomed and would advance the program.

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