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Arithmetic Quantum Gravity Without Singularities

A Categorical Reading of the Conformal Primon Gas

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1. INTRODUCTION

One of the most striking results in recent mathematical physics is the discovery by Hartnoll and Yang [12] that near a spacelike singularity, gravity becomes *arithmetic*. Their paper, "The Conformal Primon Gas at the End of Time," establishes a chain of connections linking cosmological singularities to number theory:

BKL dynamics $\longrightarrow$ Hyperbolic billiards on $SL(2,\mathbb{Z})$
$\longrightarrow$ Automorphic waveforms (Maaß forms)
$\longrightarrow$ Automorphic L-functions
$\longrightarrow$ Primon gas labelled by primes

Each step in this chain is mathematically precise: the BKL simplification of Einstein's equations near a singularity [2] maps spatial dynamics to a particle bouncing in half the fundamental domain of $SL(2,\mathbb{Z})$ on the hyperbolic upper half-plane [4, 1]; Wheeler–DeWitt quantisation yields eigenstates that are odd automorphic forms of $SL(2,\mathbb{Z})$; each eigenstate defines an automorphic L-function via a Dirichlet series with Hecke-multiplicative coefficients [15, 3]; and the Euler product of this L-function can be interpreted as the partition function of a gas of non-interacting charged bosonic oscillators, one per prime, generalising Julia's primon gas construction [13]. The averaged partition function produces the Witten index of a fermionic primon gas, connecting to the Sato–Tate distribution and the asymptotic distribution of primes.

The present paper asks a question that Hartnoll and Yang do not address, because it lies outside the scope of their construction:

> *If the arithmetic structure of gravity (hyperbolic geometry, modular invariance, L-functions, prime-labelled spectral data) is already present in the categorical kernel of reality, does one need a singularity to access it?*

The *Panta Rhei* research program [6, 7, 8, 9, 10] develops Category τ: a mathematical framework specified by five generators $(\alpha, \pi, \gamma, \eta, \omega)$, one progression operator ρ, and seven axioms (K0–K6). Its mathematical layer (Books I–III) derives prime polarity, L-functions, the Euler product, hyperbolic geometry, and the critical-line property of τ-L-functions from the kernel axioms, without any reference to gravity, singularities, or semiclassical quantisation. Its physics layer (Books IV–V) then reads out gravity as a boundary-character identity in the holonomy algebra $H_\partial[\omega]$ — again without singularities, since the No-Singularity Theorem (V.T103 **[τ-Effective]** [10]) proves that no element of $H_\partial[\omega]$ diverges.

The purpose of this paper is threefold.

(i) To present, with full respect for the Hartnoll–Yang construction, a structural comparison showing that the same mathematical objects (hyperbolic geometry, $SL(2,\mathbb{Z})$ invariance, L-functions, Euler products, prime-labelled spectra) arise in Category τ from categorical axioms rather than BKL dynamics.

(ii) To show that in the τ-framework, the critical-line property of automorphic L-functions is a *derived* result (III.T19, **[Conjectural]** conditional on the determinant representation O3 [8]), not a standalone conjecture — shifting the Riemann hypothesis from an external assumption to a structural consequence of the framework modulo one explicit obligation.

(iii) To propose that the "end of time" at a singularity, where Hartnoll and Yang discover arithmetic structure, has a categorical counterpart — the coherence horizon — where the same arithmetic is present not because geometry breaks down, but because it was always there.

Epistemic caveat. The same asymmetry that applied to our comparison with Pinčák et al. [5] applies here. Hartnoll and Yang's paper is a serious, well-crafted work from the Department of Applied Mathematics and Theoretical Physics at Cambridge, operating within well-established formalisms (BKL dynamics, WDW quantisation, automorphic forms, Hecke theory). Category τ is developed in a seven-volume monograph series that has not yet undergone independent peer review, though its formal layer (TauLib) is machine-checked in Lean 4 [11]. All results from Category τ cited here carry the scope label **[τ-Effective]**. This paper should be read as a conditional comparison: if Category τ is granted as internally valid, what structural reading of the Hartnoll–Yang discoveries does it yield?

Notation. We write $\iota_\tau = 2/(\pi + e) \approx 0.3413$ for the master constant; $\mathbb{L} = S^1 \vee S^1$ for the lemniscate boundary; $H_\partial[\omega]$ for the boundary holonomy algebra; $\zeta_\tau(s)$ for the split-complex zeta function; and H_∞ for the universal operator on the boundary. Results from the Panta Rhei monographs are cited by registry identifier. *Notational warning:* Hartnoll and Yang use π_τ for the BKL superspace momentum conjugate to the volume parameter τ; this must not be confused with the generator π of Category τ, which is one of the five kernel generators $(\alpha, \pi, \gamma, \eta, \omega)$. The two uses of π and τ are unrelated.

2. THE HARTNOLL–YANG FRAMEWORK: A SUMMARY

We present a faithful summary of the Hartnoll–Yang construction, stated in the authors' own terms.

2.1 From BKL dynamics to hyperbolic billiards

Near a spacelike singularity, the BKL analysis [2] shows that the spatial metric at each point decouples and evolves as a Kasner-like solution punctuated by "bounces" off exponential potential walls. In the Iwasawa decomposition, the three scale factors β_a satisfy a Hamiltonian constraint (Hartnoll–Yang Eq. 2) that, after a change of variables to Milne-model coordinates (x, y) with $y > 0$, becomes

$$\pi_\tau^2 = y^2(\pi_x^2 + \pi_y^2), \qquad 0 \le x \le \tfrac{1}{2}, \quad x^2 + y^2 \ge 1. \tag{1}$$

This is the Hamiltonian for a free particle on the upper half-plane $\mathbb{H}$ with the Poincaré metric, confined to half the fundamental domain of $SL(2,\mathbb{Z})$. The $SL(2,\mathbb{R})$ symmetry of $\mathbb{H}$ is the $SO(1,2)$ Minkowski symmetry of the original superspace metric.

We note, as Hartnoll and Yang themselves emphasise (their footnote 1), that the regime of validity of BKL-like decoupling of spatial points within the fully nonlinear and inhomogeneous evolution of Einstein's equation is not yet fully established, though positive numerical and rigorous results exist [1]. The entire construction rests on this BKL simplification being valid near the singularity.

2.2 Wheeler–DeWitt quantisation and Maaß forms

Wheeler–DeWitt quantisation promotes momenta to differential operators, yielding an eigenvalue problem on $\mathbb{H}$:

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi_k(x,y) = \left(\tfrac{1}{4} + \varepsilon_k^2\right)\Psi_k(x,y), \tag{2}$$

with Dirichlet boundary conditions on the walls of the half-fundamental domain. The solutions are the odd Maaß cusp forms of $SL(2,\mathbb{Z})$ — automorphic forms with discrete "energies" ε_k and Fourier coefficients c_n^k determined by Hecke operators T_p [3, 15].

2.3 Automorphic L-functions

The Hecke multiplicativity of the Fourier coefficients allows assembly into an L-function via a Dirichlet series and Euler product:

$$L_k(s) = \sum_{n=1}^{\infty} \frac{c_n^k}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - c_p^k p^{-s} + p^{-2s}}. \tag{3}$$

The prime coefficients are parametrised by angles: $c_p^k = 2\cos\theta_p^k$. The associated xi function $\xi_k(s)$ has the reflection symmetry $\xi_k(s) = -\xi_k(1-s)$, and its nontrivial zeros are believed to lie on the critical line $s = \frac{1}{2} + it_n^k$ — the Riemann hypothesis for automorphic L-functions.

Hartnoll and Yang show (their §4.3) that the wavefunction in the dilatation basis is proportional to the L-function along the critical line:

$$\phi_k(t) \propto L_k(\tfrac{1}{2} + it), \tag{4}$$

so that the wavefunction *vanishes at exactly the nontrivial zeros* of the L-function. This realises, within BKL quantum cosmology, the Berry–Keating–Connes programme linking Riemann zeros to the spectrum of a physical operator.

2.4 The conformal primon gas

On the real axis, the Euler product representation of $L_k(s)$ can be interpreted as the partition function of a gas of non-interacting charged bosonic oscillators, one per prime p, with Hamiltonian $H^{(p)} = \omega_p(a_p^\dagger a_p + b_p^\dagger b_p)$, energy $\omega_p = \log p$, and imaginary chemical potential $i\theta_p^k/s$ (Hartnoll–Yang Eq. 78–80). This "conformal primon gas" generalises Julia's primon gas [13] to automorphic L-functions.

Averaging the logarithm of the partition function over the chemical potentials θ_p^k using the Kesten–McKay distribution yields the Witten index of a *fermionic* primon gas, whose single-fermion energy is $\omega_p = \log p$ and whose pth fermion appears with degeneracy $(p-1)/2$ (Hartnoll–Yang Eq. 92–93). The divergence of the averaged partition function as $s \to \frac{1}{2}$ is controlled by the prime counting function $\pi(x) \sim x/\log x$, connecting the primon gas thermodynamics directly to the distribution of primes.

3. CATEGORY τ: ARITHMETIC STRUCTURE FROM AXIOMS

We now present the τ-framework's native arithmetic structure, showing how the same mathematical objects discovered by Hartnoll and Yang at the singularity arise from categorical axioms without any reference to gravity, spacetime, or singularities.

3.1 Prime polarity: primes as native structure

In the Hartnoll–Yang framework, primes enter as labels for the oscillators of the primon gas — they are the indices over which the Euler product runs. The primes themselves are taken as given (from classical number theory).

In Category τ, primes are *earned* from the kernel axioms. The Hyperfactorisation Theorem (I.T04 **[τ-Effective]** [6]) proves that every τ-object has a unique canonical normal form via three critical lemmas (tetration injectivity, no-tie determinism, strict remainder descent). The Prime Polarity Theorem (I.T05 **[τ-Effective]** [6]) then shows that every prime carries a canonical bipolar polarisation via γ/η dominance: each prime is classified as B-dominant (γ-heavy) or C-dominant (η-heavy), inducing a global spectral structure with finite local witnesses.

This means the prime-labelled spectral data that Hartnoll and Yang discover at the endpoint of their BKL $\to$ billiards $\to$ Maaß forms $\to$ L-functions chain is already present at the *foundation* of Category τ, in Book I, before any geometry or physics is introduced.

3.2 The lemniscate boundary: a hyperbolic spectral space

The fundamental domain of $SL(2,\mathbb{Z})$ on the hyperbolic upper half-plane is the stage on which Hartnoll and Yang's automorphic waveforms live. In Category τ, the analogous object is the lemniscate boundary $\mathbb{L} = S^1 \vee S^1$, defined algebraically as the bipolar spectral algebra (I.D18 **[τ-Effective]** [6]):

$$\mathbb{L} \;=\; A_\tau^{(B)} \times A_\tau^{(C)}, \tag{5}$$

with crossing point ω and polarity involution σ. The two lobes correspond to the B-polarised and C-polarised spectral sectors; the crossing point is the single generator ω where the polarities meet.

The lemniscate is a *hyperbolic* object, but in a different mathematical sense from the upper half-plane, and this difference must be stated precisely.

In the Hartnoll–Yang framework, the upper half-plane $\mathbb{H}$ carries the Poincaré metric with $SL(2,\mathbb{R})$ isometry group, and automorphic forms satisfy the *Laplace equation* $\Delta\Psi = \lambda\Psi$ (an elliptic PDE: no characteristics, isotropic propagation, maximum principle holds).

In Category τ, the lemniscate $\mathbb{L}$ carries the split-complex structure $j^2 = +1$, and τ-holomorphic functions satisfy the *wave equation* $\Box f = 0$ (a hyperbolic PDE: two families of characteristics, directional propagation, no maximum principle; II.T44 **[τ-Effective]** [7]). The algebraic idempotent decomposition $f = e_+ f_+ + e_- f_-$ *is* the decomposition into characteristic families: the e_+-component propagates along one null direction, the e_--component along the other (III.D41 **[τ-Effective]** [8]).

These are genuinely different function theories. The structural correspondence is therefore not an identification (Maaß forms on $\mathbb{H} \neq \tau$-holomorphic functions on $\mathbb{L}$), nor a duality in any formal categorical sense, but a *structural analogy*: both frameworks use a hyperbolic-signature space (Poincaré metric in one case, split-complex algebra in the other) and extract prime-labelled spectral data from functions on that space (Hecke eigenvalues in one case, bipolar spectral characters in the other). The common thread is hyperbolic signature, not identical function theory. The arithmetic content — primes, L-functions, Euler products — is carried by both, but through different analytic mechanisms.

3.3 L-functions as spectral determinants

In the Hartnoll–Yang framework, L-functions arise from the Hecke structure of automorphic forms: the multiplicativity of Fourier coefficients allows assembly into a Dirichlet series with an Euler product.

In Category τ, L-functions arise differently. The universal operator H_∞ acts on the Hilbert space $L^2(\mathrm{Char}(\mathbb{L}))$ of boundary characters (III.D16 **[τ-Effective]** [8]). At τ-effective cutoff N:

$$L_{\leq N}(s, \varpi) = \det\big(I - s^{-1} H_{\varpi, \leq N}\big), \qquad (6)$$

where ϖ denotes a cuspidal automorphic representation (III.D32 **[τ-Effective]** [8]); we use ϖ rather than the standard π to avoid confusion with the τ-kernel generator π (see the notational warning in §1). The Riemann zeta, Dirichlet L-functions, and Hecke L-functions are all special cases of this unified construction.

The Euler product is not an observed property of a particular number-theoretic function; it is the *categorical Chinese Remainder Theorem lifted to the holomorphic level*:

Proposition 3.1 (Adelic Euler Product [8, III.P07]). *Every τ-holomorphic function on the τ-adele ring $\mathbb{A}_\tau$ decomposes into local factors at each prime* ***[τ-Effective]****. This is the adelic form of the Euler product.*

The construction proceeds in three steps [8]:

(a) **Algebraic CRT (III.T10).** The canonical ring isomorphism $\mathbb{Z}/\mathrm{Prim}(k)\mathbb{Z} \xrightarrow{\sim} \prod_{i=1}^{k} \mathbb{Z}/p_i\mathbb{Z}$ decomposes the primorial ring into independent local factors at each prime, with inverse given by idempotent assembly. This is purely τ-internal: no subtraction, no signed arithmetic, no extended Euclidean algorithm.

(b) **Functorial lift (III.D20, Reconstruction Functor).** The CRT isomorphism lifts to an equivalence of module categories: $\mathcal{R}_k : \prod_{i=1}^{k}(R_i\text{-Mod}) \xrightarrow{\sim} R\text{-Mod}$. Applied to holomorphic function spaces, this yields a decomposition of the spectral algebra into local factors: $\mathrm{Hol}(\mathbb{A}_\tau) \cong \prod_p{}' \mathrm{Hol}_p(\mathbb{Z}_p^\tau)$, where the restricted product condition reflects that local components are unramified (lying in the maximal compact subgroup) at almost all places — matching the standard adelic restricted-product condition.

(c) **Convergent product (III.P07).** The Central Theorem (II.T40 [7]) guarantees that holomorphic structure on a τ-object is determined by its spectral algebra on the boundary $\mathbb{L}$, and the CRT decomposes this spectral algebra prime by prime. The Euler product is therefore not an observed property of a particular series but a *structural consequence* of the CRT applied at the holomorphic level.

Spectral determinant vs. Dirichlet series. An important distinction must be stated. In Hartnoll–Yang, L-functions are built from Hecke eigenvalues c_n^k via a Dirichlet series (their Eq. 14); the multiplicativity of c_n^k follows from the Hecke relations, which are properties of $SL(2, \mathbb{Z})$. In Category τ, L-functions are spectral determinants of an operator on a metric graph (the lemniscate). These are genuinely different constructions. Their identification — showing that the spectral determinant of $H_{\pi, \leq N}$ reproduces the classical Dirichlet series with Hecke-multiplicative coefficients — is precisely the content of obligation O3 (the determinant representation). Without O3, the τ-L-functions are well-defined spectral objects, but their equivalence to classical automorphic L-functions remains conjectural. The split-complex zeta function $\zeta_\tau(s)$ admits a bipolar Euler product (III.T16 **[τ-Effective]** [8]):

$$\zeta_\tau(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \mathrm{Label}(p) \cdot p^{-s}}, \qquad (7)$$

where $\mathrm{Label}(p) \in \{e_+, e_-, \text{mixed}\}$ is determined by the spectral trichotomy (III.T14 **[τ-Effective]**). The label stabilises immediately (III.T13 **[τ-Effective]**): $\text{Label}(p) = B$ (i.e., e_+) if $p \equiv \pm 1 \pmod 8$; $\text{Label}(p) = C$ (i.e., e_-) if $p \equiv \pm 3 \pmod 8$; and $\text{Label}(2) = \text{mixed}$ (X-type), reflecting that the unique even prime participates in both lobes of the lemniscate. The local Euler factor at a mixed prime carries both idempotent components and does not split into a pure e_+ or e_- factor.

3.4 The critical-line property: derived, not conjectured

Hartnoll and Yang's L-functions are *conjectured* to have zeros on the critical line $\text{Re}(s) = \frac{1}{2}$ — the generalised Riemann hypothesis for automorphic L-functions. This conjecture is supported by random matrix theory (the zeros share statistical properties with eigenvalues of random matrices [14]) but remains unproven.

In Category τ, the critical-line property is a *derived result*:

Theorem 3.2 (Critical Line Theorem [8, III.T19]). *Conditional on III.T18 (the spectral correspondence via the determinant representation): self-adjointness of H_L (III.T17) forces all eigenvalues real, which via the spectral correspondence forces all non-trivial zeros of ζ_τ to lie on* $\text{Re}(s) = \frac{1}{2}$ ***[Conjectural]*** *(conditional on O3). The K5 off-diagonal exclusion is the mechanism: off-critical-line zeros would require imaginary spectral coupling, which K5 forbids.*

The honest gap is precisely identified: obligation O3 (the determinant representation $\det(sI - H_L) = C_L(s) \cdot \zeta_L(s)$) remains conjectural [8]. But the structural content is clear: *within* the τ-framework, the critical-line property follows from the axioms (specifically K5, the beacon non-successor axiom) rather than being assumed.

This means the Riemann hypothesis, which Hartnoll and Yang encounter as a deep conjecture constraining their BKL waveforms, is — from the τ-perspective — a *structural consequence* of the same axioms that generate prime polarity, the lemniscate boundary, and the spectral algebra.

3.5 The Langlands correspondence: automorphic-Galois duality

Hartnoll and Yang's L-functions are automorphic L-functions of $SL(2, \mathbb{Z})$, connected to the Langlands programme through the Hecke eigenvalue structure and the Sato–Tate conjecture for the distribution of prime angles θ_p^k.

In Category τ, the Langlands correspondence is formulated as automorphic-Galois duality on the boundary character lattice $\mathbb{Z}^2$ (III.D63–D64 **[τ-Effective]** [8]):

- The **Galois axis** (m-axis): prime-by-prime decomposition via CRT of Frobenius elements $(Fr_p)_p$.
- The **automorphic axis** (n-axis): eigenvalue decomposition of the spectral operator.

The Functoriality Theorem (III.T36 **[τ-Effective]**) and Base Change-Transfer Naturality (III.T37 **[τ-Effective]**) establish that the two axes are related by natural transformations on the enriched bi-square. The Sato–Tate distribution of prime angles, which Hartnoll and Yang invoke for the averaging over conformal primon gases, is a deep result in classical number theory. The τ-framework establishes that B- and C-type primes are equidistributed (density $\frac{1}{2}$ each, via the internal bipolar classifier and the Legendre symbol criterion $\left(\frac{2}{p}\right)$; III.T13 **[τ-Effective]** [8]), but does *not* currently derive the full $\sin^2\theta$ distribution for the Hecke eigenvalue angles from first principles. We flag this as an open obligation (O4) alongside the determinant representation (O3): the spectral measure on the boundary algebra plausibly generates this distribution, but the derivation remains to be completed.

4. POINT-BY-POINT STRUCTURAL COMPARISON

Table 1 summarises the structural correspondence.

4.1 Where the frameworks converge

The convergence is remarkable and merits explicit statement:

(i) **Gravity is arithmetic.** Both frameworks arrive at the conclusion that at a fundamental level, gravitational dynamics is governed by number-theoretic structures — L-functions, Euler products, prime-labelled spectra, modular invariance. Hartnoll and Yang reach this conclusion by pushing gravity to its extreme (the singularity); Category τ reaches it by starting from an arithmetic kernel and reading gravity out.

(ii) **The Euler product is physically meaningful.** Both frameworks interpret the Euler product not merely as a formal identity in analytic number the-

Table 1. Structural correspondence between the Hartnoll–Yang framework and Category τ on the arithmetic elements of near-singularity gravity.

Element	Hartnoll–Yang	Category τ
Hyperbolic geometry	Upper half-plane $\mathbb{H}$ with Poincaré metric; arises from BKL superspace	Lemniscate $\mathbb{L} = S^1 \vee S^1$ with split-complex ($j^2 = +1$) signature; arises from prime polarity
Modular invariance	$SL(2, \mathbb{Z})$ symmetry of the fundamental domain; constrains WDW wavefunctions to be automorphic forms	Profinite completion of boundary algebra; omega-tail compatibility conditions (I.D25) force modular structure
L-functions	Assembled from Hecke-multiplicative Fourier coefficients c_n^k via Dirichlet series	Spectral determinants of universal operator H_∞ on boundary characters (III.D32)
Euler product	Observed property of L-functions; product over primes p with local factors $L_k^{(p)}(s)$	CRT lifted to holomorphic level (III.P07); product structure is a theorem of the framework
Prime-labelled data	Primes label oscillators of the primon gas; angles θ_p^k are chemical potentials	Primes carry canonical bipolar polarisation (I.T05); "angles" are the B/C polarity ratios, structurally determined
Critical line	Conjectured: nontrivial zeros of $L_k(s)$ on $\mathrm{Re}(s) = \frac{1}{2}$ (GRH)	Derived (conditional on O3): Critical Line Theorem III.T19; mechanism: K5 off-diagonal exclusion
Sato–Tate	Conjectured: θ_p^k distributed as $\sin^2\theta$ on $[0, \pi]$ for each k	B/C equidistribution derived (III.T13); full $\sin^2\theta$ distribution is an open obligation (O4)
Singularity	Required: BKL dynamics operates near spacelike singularity; "end of time" where classical description breaks down	No singularity (V.T103): $H_\partial[\omega]$ is profinite (compact), no divergent elements; "end of time" is the coherence horizon
Quantisation	Semiclassical: WDW quantisation of classical BKL Hamiltonian; ordering ambiguity acknowledged	Not semiclassical: quantum and gravitational structures are co-originary from the kernel axioms

ory but as encoding physical structure: a primon gas partition function (Hartnoll–Yang) or a CRT-decomposed boundary algebra (Category τ).

(iii) **Primes are not external labels.** In both frameworks, primes are not arbitrary indices but carry structural content: chemical potentials θ_p^k (Hartnoll–Yang) or bipolar polarisation $\mathrm{Label}(p) \in \{e_+, e_-\}$ (Category τ).

(iv) **The critical line matters physically.** Both frameworks treat the location of L-function zeros on $\mathrm{Re}(s) = \frac{1}{2}$ as a physical statement, not merely a mathematical curiosity: it determines where the dilatation-basis wavefunction vanishes (Hartnoll–Yang) or where the spectral operator's eigenvalues lie (Category τ).

4.2 Where the frameworks diverge

(i) **Starting point.** Hartnoll and Yang start from gravity (Einstein's equations) and discover arithmetic at the singularity. Category τ starts from arithmetic (kernel axioms, prime polarity) and derives gravity as a readout. The direction of inference is reversed.

(ii) **Singularity.** The BKL analysis requires a spacelike singularity — a physical regime where classical geometry breaks down and the billiard description becomes exact. Category τ has no singularities: the No-Singularity Theorem (V.T103 **[τ-Effective]**) proves that the boundary holonomy algebra $H_\partial[\omega]$ is profinite (compact), so no element diverges.

(iii) **Status of the Riemann hypothesis.** In Hartnoll–Yang, the GRH for automorphic L-functions is a conjecture — supported by the statistical properties of zeros (random matrix universality) but unproven. In Category τ, the Critical Line Theorem (III.T19) is a derived result (conditional on the determinant representation O3).

(iv) **Semiclassical vs. categorical.** The WDW quantisation is explicitly semiclassical, with ordering ambiguities acknowledged. Category τ is not semiclassical; its "quantum" structure is native to the kernel.

(v) **Chemical potentials vs. structural polarities.** The angles θ_p^k in the primon gas are parameters — constrained by modular invariance but not uniquely determined for a given state. In Category τ, the "angles" are structural consequences of prime polarity (I.T05), determined by the CRT decomposition with no free parameters.

5. THE RIEMANN HYPOTHESIS: A DEEPER READING

Hartnoll and Yang's most evocative result is that the wavefunction in the dilatation basis vanishes at the zeros of the L-function (Eq. 4). This connects the Riemann hypothesis directly to the physics of BKL singularities: the zeros are the nodes of the quantum gravitational wavefunction at the end of time.

From the τ-perspective, this connection is not accidental but *inevitable*. The Critical Line Theorem (III.T19) establishes that the zeros lie on $\text{Re}(s) = \frac{1}{2}$ because the universal operator H_∞ is self-adjoint on the boundary character space, and K5 (the beacon non-successor axiom) forbids the off-diagonal couplings that would produce imaginary eigenvalues. The physical interpretation: the spectral purity of the boundary algebra *forces* the zeros to the critical line, not as a happy accident of arithmetic but as a structural necessity of the categorical kernel.

The Grand GRH (III.D31 **[τ-Effective]** [8]) extends this to all automorphic L-functions at the adelic level, encompassing the full scaling chain $\zeta \to$ Dirichlet $\to$ Hecke $\to$ automorphic. The Hartnoll–Yang L-functions $L_k(s)$, being automorphic L-functions of $SL(2,\mathbb{Z})$, fall within this scope.

We emphasise, however, the honest gap: the bridge between the τ-internal critical-line result and the orthodox formulation depends on obligation O3 (the determinant representation), which remains conjectural [8]. The τ-framework does not claim to have *proven* the Riemann hypothesis in the standard sense; it claims to have identified a structural reason for it within a framework where it follows from the axioms.

6. SINGULARITY VS. COHERENCE HORIZON

Hartnoll and Yang's title — "The Conformal Primon Gas at the End of Time" — refers to the spacelike singularity where the BKL dynamics operates. This is a regime where the volume of a local spatial element collapses "doubly exponentially" ($\tau \to \infty$ in their parameter), spatial gradients decouple, and the classical description of gravity breaks down. The arithmetic structure they discover is revealed precisely *because* geometry fails.

Category τ offers a structurally different account of "the end of time." The No-Singularity Theorem (V.T103 **[τ-Effective]** [10]) proves that:

- The boundary holonomy algebra $H_\partial[\omega]$ is profinite (compact), so no element diverges.
- The source term $T[\chi_n]$ is bounded at every depth.
- The α-orbit has a first element α_1 with no sub-Planckian descent — there is no "before the beginning."

The τ-analogue of the "end of time" is the *coherence horizon*: the accumulation point α_ω of the α-orbit as $n \to \infty$. At this point, three things converge **[τ-Effective]** [10]:

- The defect entropy $S_{\text{def}} \to 0$: all transient excitations have been absorbed.
- The refinement entropy $S_{\text{ref}} \to S_{\text{total}}$: the full entropy budget is accounted for by refinement, not defect.
- The absorbing pattern $\mathcal{P}_\infty$ reaches a stable attractor: boundary characters circulate perpetually on the lemniscate $\mathbb{L}$ in a self-sustaining coherent flow, with no further generative activity.

This is a *saturation*, not a singularity. The universe does not collapse or diverge; it completes its finite generative unfolding and enters an eternal circulation. The arithmetic structure is present at the coherence horizon because it was present from the beginning — in the kernel axioms that generated prime polarity and the spectral algebra.

The crucial observation is that the arithmetic structure — primes, L-functions, Euler products, modular invariance — is present at the coherence horizon not because geometry breaks down there, but because it was present *from the beginning*, in the categorical kernel. Hartnoll and Yang discover this structure at the singularity because the singularity strips away the geometric complexity and reveals the underlying arithmetic. From the τ-perspective, the same stripping could have been accomplished by starting from the axioms.

This has a philosophical consequence that we state explicitly: if the arithmetic structure of gravity is fundamental (not emergent at singularities), then the Hartnoll–Yang construction is not discovering *new* physics at the end of time — it is *rediscovering* the arithmetic kernel that was always there, made visible by the simplification of the BKL regime. This is not a criticism; it is an endorsement. Their discovery is genuine precisely because the structure they find is real. The question is only whether that structure requires a singularity to be accessed, or whether it is accessible from categorical first principles.

7. THE PRIMON GAS IN CATEGORICAL LANGUAGE

Hartnoll and Yang's conformal primon gas provides a statistical-mechanical interpretation of automorphic L-functions: the Euler product is the partition function of a gas of non-interacting oscillators labelled by primes. In Category τ, this construction admits a natural categorical *re-reading*.

Scope of this section. We emphasise at the outset that what follows is a *structural re-reading*, not a derivation. We do not claim to derive the Hartnoll–Yang primon-gas Hamiltonian, its partition function, or its averaging procedure from the τ-axioms; nor do we claim to reproduce the Witten-index computation as a theorem of the framework. What we offer is a translation: given that the primon-gas construction stands on its own mathematical ground (Hartnoll–Yang [12], Julia [13]), we identify which categorical objects in Category τ correspond to which elements of that construction. The value of the translation, if any, lies in making explicit a shared structural vocabulary, not in offering an alternative proof of the primon-gas results.

The spectral algebra $A_{\text{spec}}(\mathbb{L})$ (II.D60 **[τ-Effective]** [7]) decomposes via the CRT (III.T10 **[τ-Effective]**) into local factors at each prime:

$$A_{\text{spec}}(\mathbb{L}) \;\cong\; \prod_{p\in\mathbb{P}} A^{(p)}_{\text{spec}}(\mathbb{L}). \tag{8}$$

Each local factor $A^{(p)}_{\text{spec}}(\mathbb{L})$ carries the data of a single prime — its polarity (B or C dominant), its spectral weight, and its contribution to the Euler product. This is the categorical counterpart of a single primon oscillator.

The "chemical potential" θ^k_p in the Hartnoll–Yang primon gas is, in categorical language, the *phase of the bipolar spectral character* at prime p — the angle between the B-component and the C-component of the boundary character restricted to the p-local factor. In Category τ, this angle is structurally determined by the prime polarity theorem (I.T05) and the CRT decomposition; it is not a free parameter.

The averaging over chemical potentials that Hartnoll and Yang perform (using the Kesten–McKay distribution, their Eq. 88) has a categorical interpretation: it is the spectral measure on the ensemble of boundary characters, weighted by the profinite structure of the boundary algebra. The result — the Witten index of a fermionic primon gas — corresponds to the alternating sum over sector contributions in the boundary algebra, which is related to the Möbius function $\mu(n) = (-1)^k$ for n a product of k distinct primes.

This last connection is especially striking. Hartnoll and Yang note (their footnote 3) that the Möbius function "might be thought of as the mathematical discovery of fermions in 1832." This observation is well-established in the primon gas literature going back to Julia [13]. The τ-framework's specific contribution, if any, would be to show that the fermionic nature is not merely an analogy but a *structural consequence* of prime polarity: the alternating sign $\mu(n) = (-1)^k$ for n a product of k distinct primes reflects the bipolar parity of the CRT decomposition, where each prime factor contributes one polarity flip ($e_+ \leftrightarrow e_-$) in the boundary algebra. Whether this structural observation constitutes a genuine advance over the classical number-theoretic fact — or merely restates it in categorical language — is a question we leave to the reader.

8. DISCUSSION

8.1 What Hartnoll and Yang do well

Several features of the Hartnoll–Yang paper deserve explicit recognition:

(i) **A genuine discovery.** The chain from BKL dynamics through automorphic forms to primon gases is a real mathematical achievement, not a formal exercise. The connection between cosmological singularities and number theory is non-obvious and physically meaningful.

(ii) **The wavefunction-zeros correspondence.** The result that the dilatation-basis wavefunction vanishes at the zeros of the L-function (Eq. 4) is a concrete realisation of the Berry–Keating–Connes programme, achieved within a physically motivated context (quantum cosmology) rather than an abstract operator-theoretic one.

(iii) **The primon gas interpretation.** The identification of the Euler product as a partition function, with imaginary chemical potentials constrained by modular invariance, provides a statistical-mechanical language for automorphic L-functions that may prove fruitful beyond the BKL context.

(iv) **The averaging procedure.** The use of the Kesten–McKay distribution to average over conformal primon gases, yielding the Witten index of a fermionic gas, is an elegant application of random matrix theory to extract universal features from a family of modular-invariant partition functions.

(v) **Future directions.** The paper's Section 7 identifies several genuinely interesting avenues: adelic perspectives, p-adic holography, period functions, many-body CQM, and superspace holography. These are not idle speculations but well-motivated extensions of the paper's core results.

8.2 What Category τ adds

(i) **Singularity-independence.** The arithmetic structure Hartnoll and Yang discover at the singularity is, in the τ-framework, present at the foundation — not revealed by extreme conditions but *constitutive* of the mathematical world. This suggests that the BKL regime is a *window* into the arithmetic kernel, not the kernel's *origin*.

(ii) **A structural account of the Riemann hypothesis.** The transition from "the GRH is a deep conjecture constraining our wavefunctions" to "the critical-line property is a theorem of the framework" is a significant structural shift. Even though the bridge (O3) remains conjectural, the identification of a *mechanism* (K5 off-diagonal exclusion) for the critical-line property is an advance over regarding it as a brute fact.

(iii) **No ordering ambiguity.** The WDW quantisation in Hartnoll–Yang has an ordering ambiguity (acknowledged in their §2.1); the choice of DeWitt ordering is natural but not unique. Category τ's spectral operator H_∞ is defined on the boundary algebra without quantisation and therefore without ordering ambiguity.

(iv) **Prime angles as structural invariants.** The chemical potentials θ_p^k in the primon gas are parameters of the theory (one per prime per energy level). In Category τ, the corresponding data — bipolar polarity ratios — are determined by the CRT decomposition and carry no free parameters.

8.3 What Category τ does not provide

We acknowledge explicitly a structural asymmetry in this comparison. Hartnoll and Yang provide a rich *dynamical* narrative: the spatial metric undergoes Kasner epochs punctuated by bounces; the billiard trajectory in $\mathbb{H}$ is chaotic; the wavefunction evolves as $e^{-\tau/2+i\varepsilon_k\tau}$ toward the singularity; the primon gas has a temperature and a specific heat. Category τ provides *structural* architecture (axioms generating arithmetic objects) but no comparable dynamics. There is no τ-analogue of a Kasner epoch; no billiard bounces; no time-dependent wavefunction in the BKL sense.

This is a genuine limitation of the present comparison, though it may reflect the different levels at which the two frameworks operate: Hartnoll–Yang work at the level of solutions to Einstein's equations in a specific regime; Category τ works at the level of the categorical kernel from which the equations themselves are read out. Whether the BKL dynamics can be *recovered* from the τ-framework as a chart-level readout of the boundary algebra in a strong-curvature regime is an open question that lies beyond the scope of this paper.

8.4 The aggregate weight of obligation O3

Before enumerating open questions, we collect in one place the full load that obligation O3 (the determinant representation $\det(sI - H_L) = C_L(s)\cdot\zeta_L(s)$) carries across this paper, so that the reader can weigh the conditionality of our claims without having to reassemble it from scattered footnotes.

Three distinct statements in this paper are conditional on O3:

(1) the identification of τ-L-functions (spectral determinants of H_ϖ, Eq. 6) with classical automorphic L-functions (Dirichlet series with Hecke-multiplicative coefficients, §3.3);

(2) the Critical Line Theorem III.T19 (§3.4, Theorem 3.2), and hence the structural derivation of the Riemann hypothesis within the τ-framework

(§5); and

(3) the main comparative claim of Table 1 that the critical-line property is "derived" rather than "conjectured" in Category τ.

O3 is therefore not a peripheral technical gap; it is the single hinge on which the paper's strongest claims turn. We make no attempt in this paper to close O3 — its proof is a research-program-level obligation of Book III [8]. A reader who does not grant O3 should read the τ-framework's contribution here as: *a structurally motivated candidate mechanism for the Riemann hypothesis (the K5 off-diagonal exclusion acting on a self-adjoint operator on the boundary character space), contingent on an explicitly stated determinant representation.* A reader who does grant O3 should read it as: *within Category τ, the critical-line property is a theorem, and its physical appearance in the Hartnoll–Yang construction is a window onto the same categorical kernel.* The paper's structure is designed so that both readings are coherent.

8.5 What remains open in both frameworks

(i) **The determinant representation (O3).** Both frameworks encounter a gap when connecting L-function zeros to spectral data. Hartnoll and Yang assume the GRH; Category τ derives it conditionally on O3. Neither has closed the gap fully. See §8.4 for the full load O3 carries in this paper.

(ii) **The many-body problem.** Hartnoll and Yang note (§7.3) that the full theory must tensor automorphic states at each spatial point, leading to a many-body CQM. Category τ has a corresponding open question: how the boundary algebra's spectral structure interacts across different base points on τ^1.

(iii) **Observational access.** Neither framework currently makes predictions accessible to observation. The BKL regime is approached only at singularities (Big Bang, black hole interiors); the coherence horizon is a structural concept without a direct observable proxy. The most promising bridge may be CMB signatures of near-singularity dynamics, which Hartnoll and Yang gesture toward and which the τ-framework addresses through its CMB pipeline (V, Parts I–II [10]).

9. CONCLUSION

Hartnoll and Yang have discovered that near a spacelike singularity, gravity becomes arithmetic: the BKL dynamics maps onto hyperbolic billiards whose quantum states are automorphic forms, whose L-functions admit Euler products over primes, and whose partition functions describe a conformal primon gas. This is a genuine and important discovery.

We have shown that the mathematical structure they discover — hyperbolic geometry, modular invariance, L-functions, Euler products, prime-labelled spectral data, and the critical-line property (the last conditional on the determinant representation O3) — arises natively within Category τ from categorical axioms, without requiring a spacetime singularity or semiclassical quantisation. The convergence between the two frameworks is deep, though the precise content of the shared slogan "gravity is arithmetic" differs:

- For Hartnoll and Yang, it means: *near a spacelike singularity, the classical dynamics of gravity maps onto arithmetic quantum chaos on the fundamental domain of $SL(2, \mathbb{Z})$, and the quantum states are automorphic forms with number-theoretic properties.* This is a precise mathematical statement about a specific physical regime.
- For Category τ, it means: *the categorical axioms that generate prime polarity also generate gravity as a boundary-character identity; the arithmetic and the gravity share a common origin in the kernel.* This is a structural claim about the architecture of the framework.

These are not identical statements. The first is a dynamical discovery about what happens to gravity at extremes; the second is an architectural claim about what generates both gravity and arithmetic. The convergence is that both arrive at the same mathematical objects; the divergence is in the direction of inference — Hartnoll and Yang discover arithmetic by pushing gravity to its extreme; Category τ discovers gravity by starting from an arithmetic kernel.

The most striking implication is this: if the arithmetic structure of gravity is not *created* at singularities but *revealed* there — if it is constitutive of reality rather than emergent at its extremes — then the Hartnoll–Yang construction is not merely a result about cosmological singularities. It is a window into the categorical kernel of physics itself.

We close by echoing the spirit of Hartnoll and Yang's own conclusion: the emergent automorphic and conformal symmetries of near-singularity gravity may indeed "guide the search for a complete holographic description of the end of time." Category τ suggests that this search might begin not at the end of time, but at the beginning of mathematics.

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